Cash-flow Risk, Discount Risk, and the Value Premium

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May 10, 2005

Abstract

We propose a general equilibrium model with multiple assets able to match both the time series and the cross-sectional predictability of stock returns. The cross section of average returns is determined by both cash-flow risk and discount risk. We show that if cross-sectional differences in average returns are mainly determined by discount risk, then a counterfactual prediction obtains: Assets with high values normalized by cash-flows — growth stocks — command a high premium relative to value stocks. Using our model, we find that a strong cross-sectional variation in cash flow risk is necessary to obtain a quantitatively plausible value premium. In addition, general equilibrium restrictions on the market portfolio also generate the value premium puzzle, that is, the inability of the CAPM to explain the cross section of average returns of price sorted portfolios. We use the model to interpret the Fama and French (1993) model, and, in particular, to offer an interpretation of the HML factor.

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I. INTRODUCTION

Historically, stocks with high book-to-market ratios, value stocks, have yielded higher average returns than stocks with low book-to-market ratios, growth stocks.\(^1\) The CAPM's major failure is its inability to price book-to-market sorted portfolios. As a result of this failure, a large collection of explanations have been proposed to address this value premium puzzle. These explanations range from behavioral stories linked to overreaction of investors to firm performance, as in Lakonishok, Shleifer and Vishny (1994),\(^2\) to rational asset pricing models where the value premium is linked to compensation for risk, as in Fama and French (1993).\(^3\) However, regardless of whether the story is a behavioral or a rational one, the proposed model will have to yield plausible magnitudes for the cross-section and time series predictability. Here, unfortunately, the financial literature is lagging, both for behavioral and rational stories. In this paper, we propose a general equilibrium, rational asset pricing model that addresses this question, and obtains quantitative predictions for the time series and the cross-section of stock returns.

In particular, we consider an economy with multiple assets that differ in their cash-flow risk, that is, in the covariance of their cash-flows with the aggregate economy, and that have time varying expected dividend growth. The economy is one where a representative agent displays time varying risk preferences through a habit specification. Our model shows that stocks' expected excess returns depend both on cash-flow risk and discount risk, that is, the sensitivity of returns to changes in aggregate risk preferences. We find that (a) substantial cross-sectional differences in cash-flow risk are necessary to generate a plausible value premium; (b) the unconditional CAPM fails, and thus a value premium puzzle arises, because of general equilibrium restrictions on the total wealth portfolio; and (c) an HML factor lines up returns as it captures aggregate differences in cash flow risk in the economy. In addition, the model also sheds light on the performance of the conditional CAPM models that have been recently proposed in the empirical financial literature.

Specifically, we borrow the cash-flow process of Menzly, Santos and Veronesi (2004, MSV henceforth).\(^4\) As mentioned, firms' cash-flows feature both time varying expected dividend growth.

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\(^1\) Rosenberg, Reid, and Lanstein (1985) and Fama and French (1992).
\(^2\) See also DeBondt and Thaler (1987).
\(^3\) See also the conditional asset pricing models of Jagannathan and Wang (1996), Ferson and Harvey (1999), Lettau and Ludvigson (2001), and Santos and Veronesi (2005).
\(^4\) To the best of our knowledge the first fully fledged general equilibrium model of the cross section of stock returns is Gomes, Kogan, and Zhang (2003) who build on the partial equilibrium model of Berk, Green and
growth, as well as cross-sectional differences in cash-flow risk. In a general equilibrium setting, however, there is an add-up constraint, as, by definition, the variance of the aggregate endowment growth process must be a weighted average of its covariances with individual assets’ cash-flow growth. This general equilibrium restriction both limits the degrees of freedom, as well as induces some patterns in aggregate consumption that partly explain the value premium puzzle.

The second ingredient of our model is the stochastic discount factor we employ. Ours is a representative agent economy where preferences are of the external habit persistence type introduced by Campbell and Cochrane (1999). As it is well known these preferences are useful in generating plausible quantitative implications for the market portfolio. Specifically, our model is a generalization of the habit persistence model of MSV that allows for closed form solutions for prices and returns, which helps greatly in the interpretation of the results. This model generates discount effects that can be calibrated to match the time series properties of the aggregate market data. Given these quantitatively plausible discount effects we can then investigate the strength of the cash-flow effects that are needed to generate the patterns in the cross section. Our model is one where there is no “book” and instead we normalize prices by cash-flows: The empirical regularities are very similar independently of whether one normalizes by one or the other.5

One advantage of our modeling strategy is that we can shut down either discount effects or the cross sectional dispersion in cash-flow risk and assess how each contributes to the cross section of average returns. Consider first the case where all assets have identical cash flow risk.6 In this case, we show that whether an asset has a high or low premium depends on the asset’s duration, by which we mean whether it has low or high expected cash-flow growth. Assets with high expected cash-flow growth are relatively more sensitive to shocks in the aggregate discount than otherwise identical assets with low expected cash-flow growth.7 When only discount effects are present then, the risk-return trade-off is only determined by the timing of Naik (1999). See also Zhang (2005).

5To compare the properties of book-to-market, $BE/ME$, sorted portfolios and cash-flow-to-price, $C/P$, sorted portfolios the reader can turn to Table II of Fama and French (1996, page 61). The same increasing relation, almost to the point, between $BE/ME$ and average excess returns obtains if instead one uses $C/P$.

6In a general equilibrium setting, all assets have identical cash flow risk only if their covariance of dividends with consumption equals the variance of consumption. That is, all assets have a cash flow risk that equals the one of the aggregate endowment itself.

7The parallel with the standard intuition in fixed income is helpful here: the price of a zero discount bond of longer maturity is more sensitive to shocks in the aggregate discount than one with shorter maturity.
cash-flows, that is by the asset’s duration. Can discount effects generate the value premium? No. Assets with strong expected cash-flow growth have high price-dividend ratios and, as just mentioned, a high sensitivity to changes in the aggregate discount, and thus a higher expected return. Thus, a counterfactual positive relation obtains between price-dividend ratios and average excess returns. Thus, if the value premium is to obtain, cash-flow effects have to be strong enough to undo discount effects.8

Indeed, consider now the case of a low duration asset whose cash-flow growth is strongly positively correlated with the growth rate of the aggregate economy. In this case, and due to its low expected dividend growth, the total value of this asset is mainly determined by the current level of cash-flows, rather than by those in the future. The price of the asset is then mostly driven by cash-flow shocks and the fundamental risk embedded in these cash-flows drives also the risk of the asset. Thus, when cash-flows display substantial fundamental risk, the asset’s premium is higher when the duration is lower. Can these cash-flow effects generate the value premium? Yes. Assets with high cash-flow risk and low duration have low price-dividend ratios. This is due to both the fact that they are risky, and thus prices have to be low to compensate agents for the risk they take, and because they have low expected dividend growth. Thus, potentially, the value premium can now arise.

Whether a value premium arises or not though depends on how the tension between “discount effects” (high risk when the asset has a high duration) and “cash-flow effects” (high risk when the asset has low duration) plays quantitatively. To assess this we simulate an economy with 200 assets over 10,000 years of quarterly data. Throughout we mimic the procedure employed in the literature of sorting assets into decile portfolios formed on the basis of price-dividend ratios. We calibrate the discount parameters to match the time series properties of the aggregate market portfolio and thus generate quantitatively relevant discount effects. We then calibrate the cash-flow effects to obtain properties of the cross section of stock returns that match well those in the data. In order to do so, we assume that stocks have cross-sectional cash flow risk that are uniformly dispersed. In our model, we find that in order to obtain the value premium some of the underlying assets need to have cash-flow growth

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8Recently Lettau and Wachter (2005) have proposed a model where firms solely differ in the timing of their cash-flows, that is in their duration, and where the economy is subject to discount shocks. Their model allows for the possibility of investor sentiment shocks that are totally uncorrelated to shocks in the aggregate endowment process. Growth stocks, which pay far in the future, are more sensitive to shocks in investor sentiment but investors do not fear these shocks and thus the lower premium growth stocks command.
processes that are, in absolute value, strongly correlated with consumption growth. That is, the strong discount effects that are needed to generate the standard battery of moments for the aggregate portfolio imply that also very strong cash-flow effects are needed to generate the standard battery of moments for the cross section of price sorted portfolios.

We show that these cash-flow effects result in a time-series variation in the relative risk of value versus growth stocks over the business cycle as measured by shocks to aggregate consumption. In particular, consistently with the data, value stocks are particularly risky during “bad” times. This is natural: Agents demand a relatively higher compensation for holding assets with cash-flows that covary positively with consumption growth when faced with adverse shocks. Thus cash-flow effects have a conditional effect as well. To put it differently, cross sectional dispersion in cash-flow risk results not only on cross sectional dispersion in unconditional average excess returns, but also on variation in the value premium that further reinforces the unconditional effect. Indeed we show that the value premium increases considerably during bad times.

Our model is one where the CAPM does not hold, either conditionally or unconditionally. The reason is that general equilibrium restrictions induce a mild predictability in expected consumption growth, which break the perfect correlation between the stochastic discount factor and the return of the market portfolio. We show that the CAPM performs poorly in our simulations but that the conditional CAPM, which is a misspecified asset pricing model in our setup, performs much better. The reason is well known: The conditional asset pricing model captures the increase in the relative riskiness of value stocks. But this effect can only arise if value stocks are also the assets that have high cash-flow risk unconditionally, as our simulations show. This provides a rationale as to why the two different strands of rational explanations that have been proposed to address the value premium, conditional asset pricing models such as Lettau and Ludvigson (2001) and models based on cash-flow risk, such as Parker and Julliard (2005) and Bansal Dittmar and Lundblad (2005), have been shown to be both relatively successful: They are both sides of the same coin in the presence of discount effects.

We then turn our attention to the factor model of Fama and French (1993). In particular we construct an HML factor, which by construction captures the cross sectional dispersion in cash-flow risk. Assets with a high cash-flow risk have, mechanically, a high loading on HML.
The premium on HML varies over time, as already mentioned so the Fama and French model performs well because it captures the sources of unconditional cross sectional variation in average excess returns through the load on HML as well as the sources of conditional variation through the variation in the premium on HML.

Our last empirical exercise consists of measuring the extent to which in our simulations the Fama and French (1993) model can be driven out by the characteristics model of Daniel and Titman (1997). In our framework the Fama and French model is misspecified and certainly the inclusion of the characteristic helps in explaining the cross section of average returns. In this our model simply echoes Berk’s (1995) point that in the presence of model misspecification the inclusion of characteristics linked to prices in tests of the cross section are likely to capture the component of the cross sectional variation in average returns that the proposed model leaves unexplained.

Importantly, our results depend on the extent to which value stocks are also those with high cash-flow risk. Recent research suggests that this indeed the case. Campbell and Vuolteenaho (2005) find that value stocks are the ones with a high “cash-flow beta” and show that the associated premium is higher than the premium associated with the “discount beta.” Cohen, Polk and Vuolteenaho (2003) obtain cash-flow betas by regressing firm’s profitability on the market’s profitability and find that indeed value stocks have higher regression coefficients. Similarly, Bansal, Dittmar, and Lundblad (2005) regress dividend growth on a moving average of consumption growth rates, an approach similar to the one of Parker and Julliard (2005), and find that indeed cash-flow betas are larger for value sorted portfolios.

The paper proceeds as follows: Next section introduces the model. Section III contains the model results for prices and expected returns. In particular, it discusses the source of the value premium in our setting. Section IV contains a simulation of our model and comparison to data. Section V concludes. All proofs are in Appendix.

II. THE MODEL

II.A Preferences

There is a representative investor who maximizes

\[ E \left[ \int_0^\infty u(C_t, X_t, t) \, dt \right], \]

(1)
where the instantaneous utility function is given by

\[ u(C_t, X_t, t) = \begin{cases} 
  e^{-\rho t} \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 1 \\
  e^{-\rho t} \log (C_t - X_t) & \text{if } \gamma = 1 
\end{cases} \]  

(2)

In (2), the variable \( X_t \) denotes an external habit level and \( \rho \) denotes the subjective discount rate.\(^\text{10}\) The exact specification of the external habit \( X_t \) is described below.

II.B Cash Flows

We consider an endowment economy with \( n \) financial assets, whose instantaneous dividend streams are denoted by \( D^i_t \), for \( i = 1, \ldots, n \). The aggregate endowment available for consumption at any time \( t \) is then equal to the sum of dividends.\(^\text{11}\) We assume that the consumption good is immediately perishable and non-storable, which yields the equilibrium restriction

\[ C_t = \sum_{i=1}^{n} D^i_t \]  

(3)

Equation (3) implies that specific assumptions that are made on the dividend process translate into particular dynamics for aggregate consumption. Unfortunately, even the assumption of relatively simple processes for \( D^i_t \) typically yield a process for consumption that is difficult to work with, and simplifying assumptions need to be made.\(^\text{12}\) In order to better understand the restrictions that have to hold in a general equilibrium setting and the nature of our assumptions below, it is instructive to review the nature of the difficulty, as also explained in Santos and Veronesi (2005). Let \( \mathbf{D}_t = (D^1_t, \ldots, D^n_t)' \) be the vector of dividends, and assume for instance that dividends are given by

\[ \frac{dD^i_t}{D^i_t} = \mu^i_D(\mathbf{D}_t) \, dt + \mathbf{\nu}_i \, d\mathbf{B}_t \]  

(4)

for some drifts \( \mu^i_D(\mathbf{D}_t) \), and where \( \mathbf{\nu}_i \) is a \( n \times 1 \) constant vector, and \( d\mathbf{B}_t \) is a \( n \times 1 \) vector of Brownian motions. From equation (3) and Ito’s lemma, we find that the process for aggregate

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\(^\text{10}\) On habit persistence and asset pricing see Sundaresan (1989), Constantinides (1990), Abel (1990), Ferson and Constantinides (1991), Detemple and Zapatero (1991), Daniel and Marshall (1997), Campbell and Cochrane (1999), Li (2001), and Wachter (2000). These papers though only deal with the time series properties of the market portfolio and have no implications for the risk and return properties of individual securities.

\(^\text{11}\) In Santos and Veronesi (2005) we provided a thorough discussion of the role of labor income in a framework similar to the one presented here.

\(^\text{12}\) Recently, Cochrane, Longstaff and Santa Clara (2004) managed to solve in closed form the case where \( n = 2 \), dividends are log-normally distributed, and agents are endowed with log utility. Their modeling device seems hard to generalize for \( n > 2 \) and different preferences.
consumption is then
\[ \frac{dC_t}{C_t} = \mu_c(s_t) \, dt + \sigma_c(s_t)^T \, dB_t \]  
(5)
where \( s_t = (s^1_t, \ldots, s^n_t)^T = (D^1_t/C_t, \ldots, D^n_t/C_t) \) are shares of consumption produced by dividends, and
\[ \mu_c(s_t) = \sum_{i=1}^n s^i_t \mu^i_D \quad \text{and} \quad \sigma_c(s_t) = \sum_{i=1}^n s^i_t \nu_i \]  
(6)

The main difficulty in obtaining tractable and interpretable formulas for asset prices lies in the dependence of the drift and the volatility of the consumption process on the shares \( s_t = (s^1_t, \ldots, s^n_t) \). Still, analytical formulas for asset prices can be obtained by making economically plausible assumptions on the joint processes of consumption \( C_t \) and shares \( s_t \), as advanced in MSV and Santos and Veronesi (2005). Here we follow Santos and Veronesi (2005) and make two assumptions:

**Assumption 1:** The aggregate consumption is given by
\[ \frac{dC_t}{C_t} = \mu_c(s_t) \, dt + \sigma_c^T \, dB_t \]
where
\[ \mu_c(s_t) = \bar{\pi}_c + \mu_{c,1}(s_t) \quad \text{where} \quad \mu_{c,1}(s_t) = s^i_t \, \theta_CF \]  
(7)
where \( \theta_CF = (\theta^1_{CF}, \ldots, \theta^n_{CF})^T \), and \( \sigma_c = (\sigma_c, 0, \ldots, 0)^T \). The specification of \( \theta^i_{CF} \) is explained below.

**Assumption 2:** For each \( i \), the share \( s^i_t \) follows the mean reverting process
\[ ds^i_t = \phi(\bar{\pi} - s^i_t) \, dt + s^i_t \sigma^i(s_t) \cdot dB_t \]  
(8)
where
\[ \sigma^i(s_t) = \nu_i' - \sum_{j=1}^n s^j_t \nu_j' \]  
(9)

The cash flow model (8) imposes a structure to the relative size of firms, where “size” is measured as the fraction of total output produced by a given firm. In particular, it imposes the economically plausible assumption that no firm will take over the economy, as \( s^i_t > 0 \) for all \( i \). Finally, the volatility \( \sigma^i(s_t) \) in (9) ensures that \( \sum_{i=1}^n s^i_t = 1 \) for all \( t \). It is worth remarking that although the form of the volatility \( \sigma^i(s_t) \) in (9) seems an ad-hoc formulation, it actually stems from the model (4) - (5), as it is possible to verify by Ito’s lemma.
II.C Cash Flow Risk

Finally, it is important to discuss the implication of Assumption 2 for the dividend process \( D_t = s_t^i C_t \). An application of Ito’s Lemma shows that dividends evolve according to the following process:

\[
\frac{dD_t^i}{D_t^i} = \mu_{D,t}^i dt + \sigma_D^i (s_t) dB_t
\]

where

\[
\mu_{D,t}^i = \overline{\mu}_c + \theta_{CF}^i \phi \left( \frac{s_t^i}{s_t} - 1 \right)
\]

\[
\sigma_D^i (s_t) = \sigma_c(i(s_t))
\]

In these formulas,

\[
\theta_{CF}^i = \nu_i' \cdot \sigma_c
\]

First, note that the expected dividend growth \( \mu_{D,t}^i = E_t \left[ \frac{dD_t^i}{D_t^i} \right] \) depends on the relative share \( \overline{s}^i / s_t^i \). When this quantity is low, it means that the asset’s relative contribution to total consumption is below its long term average, and thus a higher expected dividend growth is expected. MSV test this prediction in a set of industry portfolios and find strong support for it. In addition, note that the long term dividend growth of this asset is given by \( \overline{\mu}_c \), the unconditional expected return of consumption growth, as well as a parameter \( \theta_{CF}^i \), which is asset specific and it depends on the correlation of the stock shares with consumption growth.

Second, in this paper and because of the assumption of habit formation preferences, the stochastic discount factor is only driven by shocks to consumption growth. Thus, cash flow risk is measured by the covariance of dividends with consumption growth, given by

\[
\sigma_{CF,t}^i \equiv Cov_t \left( \frac{dD_t^i}{D_t^i}, \frac{dC_t}{C_t} \right) = \sigma_c \sigma_c' + \theta_{CF}^i - s_t^i \theta_{CF}^i
\]

The quantity \( \sigma_{CF,t}^i \), the conditional cash flow risk of asset \( i \), will play a prominent role in this paper. The term \( \theta_{CF}^i - s_t^i \cdot \theta_{CF} \) is parametrically indeterminate, that is, adding a constant to all \( \theta_{CF}^i \) leaves this term unaffected, as \( \sum_{i=1}^{n} s_t^i = 1 \). Thus, we also impose the identifiability restriction

\[
\sum_{j=1}^{n} \overline{s}^j \theta_{CF}^j = 0
\]

Thus, we obtain that the unconditional covariance between the cash flow of asset \( i \) and consumption is given by

\[
\overline{\sigma}_{CF} = E \left[ \sigma_{CF,t}^i \right] = E \left[ Cov_t \left( \frac{dD_t^i}{D_t^i}, \frac{dC_t}{C_t} \right) \right] = \sigma_c \sigma_c' + \theta_{CF}^i
\]
In other words, the parameter \( \theta_{CF}^i \) regulates the relative cash flow risk of individual assets, as it is linear in the unconditional covariance of dividend growth with consumption growth \( \bar{\sigma}_{CF} \). It is useful to emphasize that the benchmark level of risk of an asset is the riskiness of aggregate consumption. An asset is deemed risky (safe) if its cash flows are more (less) risky than aggregate consumption. This is a general equilibrium restriction, as by definition, the variance of consumption growth must be a weighted average of its covariances with individual dividend growth.

As a final remark on cash flows, note that the model is internally consistent: If we apply the general equilibrium restriction on the drift of the consumption process, (6), to the dividend process (10)

\[
E \left[ \frac{dC_t}{C_t} \right] = \sum_{i=1}^{n} s_i^t \mu_{D,t} = \bar{\pi}_c + s_t^t \theta_{CF},
\]

which equals (7) in Assumption 1. Note then that consumption growth is not i.i.d. but rather has some predictable components which are linked to variation in the vector of shares, \( s_t \). Still, as we show below there is very little predictability in practice as the parameters \( \theta_{CF}^i \) are very small.\(^{13}\)

II.D Habit Dynamics

As advanced by Campbell and Cochrane (1999), the fundamental state variable driving the attitudes towards risk in the habit model (1) – (2) is the surplus consumption ratio, \( S_t = (C_t - X_t) / C_t^{-1} \). In order to obtain closed form solutions for prices when there are multiple securities, MSV model the inverse surplus consumption ratio \( Y_t = S_t^{-1} \) as a mean reverting process. Unfortunately, their modeling device cannot be applied when \( \gamma > 1 \). In addition, they only obtained approximate formulas for the case where \( \theta_j = 0 \). For these reasons we opt here for a different strategy and model directly the process

\[
G_t = \left( \frac{C_t}{C_t - X_t} \right)^\gamma = S_t^{-\gamma}
\]

To model the dynamics of \( G_t \) start by noticing that one important difference with respect to the setups of Campbell and Cochrane (1999) and MSV is the fact that in our model consumption has some predictable components. Thus it is important to specify the process

\(^{13}\)Briefly, we show in simulations below that expected consumption growth fluctuates between a maximum of 2.22% and a minimum of 1.87%, a very mild variation compared to the 1.5% standard deviation of consumption growth that we assume. Indeed, consistently with the empirical evidence, predictability regressions in our simulated data produce a negligible level of predictability of consumption growth.
for $G_t$ in a way that is consistent with a time-varying expected consumption growth. To have guidance on the type of process for $G_t$, it is instructive to consider the implications for $G_t$ under the standard assumption that $X_t$ is an exponentially weighted average of past consumption levels, as in the classic papers by Constantinides (1990), and Detemple and Zapatero (1991), that is,

$$X_t = \lambda \int_{-\infty}^{t} e^{-\lambda(t-\tau)} C_\tau d\tau$$

Ito’s Lemma immediately shows that $dX_t = \lambda (C_t - X_t) dt$. Thus, an application of Ito’s Lemma to (16) yields the process

$$dG_t = [\mu_G (G_t) - \sigma_G (G_t) \mu_{c,1} (s_t)] dt - \sigma_G (G_t) \sigma_c dB_t^1$$

where $\mu_G (G_t)$ and $\sigma_G (G_t) > 0$ are complicated functions of $G_t$, provided in equations (36) and (37) in the Appendix. Equation (17) provides a strong intuition on how the time varying component of the drift rate of consumption $\mu_{c,1} (s_t)$ should enter the process for $G_t = S_t^{-\gamma}$. Namely, a higher expected consumption growth $\mu_{c,1} (s_t)$ implies a lower drift rate of $G_t$. The intuition is that an increase in the expected growth rate of consumption leads to a prediction of higher future consumption compared to current habit $X_t$ and thus a higher surplus consumption ratio $S_t$ in the future. Thus, given (16), this implies a lower expected $G_t$. In what follows, as in MSV and Campbell and Cochrane, we simplify (17) to obtain a more manageable process, but we retain the intuition that $\mu_{c,1} (s_t)$ enters the drift rate of $G_t$, in the same form as specified in (17). we assume in (17) the following drift and diffusion:

$$\mu_G (G_t) = k (\bar{G} - G_t) \quad \text{and} \quad \sigma_G (G_t) = \alpha (G_t - \lambda)$$

Notice then that the drift of $G_t$ has two components to it. The first one is a mean reversion component and captures the basic idea of habit persistence models, namely that the habit $X_t$ “catches up” with $C_t$ eventually. The second component links the drift rate of $G_t$ to the time varying component of the drift rate of consumption growth. Once again, in order to retain the habit formation formulation, it is important to assume that the coefficient that multiplies $\mu_{c,1} (s_t)$ equals the diffusion term in the process itself. As for the diffusion component, as in MSV, $\lambda \geq 1$ bounds $G_t$ from below at $\lambda$ and $\alpha > 0$ transmits the innovations in consumption growth, $dB_t^1$, to the convexity of the utility function. These assumptions allow us to obtain closed form formulas for asset prices. Note that the habit model of MSV is a special case of (17) and (18): In fact, MSV assume that $\gamma = 1$, and that consumption growth is i.i.d., an assumption that obtains here by setting $\mu_{c,1} (s_t) = 0$. 

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III. EQUILIBRIUM ASSET PRICES AND RETURNS

In this section we characterize the prices and returns associated with the economy described in the previous section (Section III.A). We also consider in more details two special cases that allow for a better intuition of the impact that discount and cash flow effects have on the cross-section of prices and returns. Our strategy to do this is standard. Given (2), the stochastic discount factor is given by

$$ m_t = e^{-\rho t} (C_t - X_t)^{-\gamma} = e^{-\rho t} C_t^{-\gamma} G_t. $$

Then we can use Ito’s Lemma, together with our assumptions on the dynamics of $C_t$ and $G_t = S_t^{-\gamma}$ to show that

$$ \frac{dm_t}{m_t} = -r_t^f dt + \sigma'_m dB_t, $$

where

$$ r_t^f = \rho + \gamma \mu_c + [\gamma + \alpha (1 - \lambda S_t^\gamma)] \mu_{c,1} (s_t) + k (1 - G S_t^\gamma) - \frac{1}{2} \gamma (\gamma + 1) \sigma_c^2 - \gamma \alpha (1 - \lambda S_t^\gamma) \sigma_c^2 $$

and the first, and only non-zero, entry in the diffusion component vector, $\sigma_m$, is given by

$$ \sigma'_m = -[\gamma + \alpha (1 - \lambda S_t^\gamma)] \sigma_c. \quad (19) $$

Then we exploit our assumptions on the dynamics of $C_t$, $G_t = S_t^{-\gamma}$ and $s_t^i$ to solve for

$$ P_t^i = E_t \left[ \int_t^\infty \left( \frac{m^\tau}{m_t} \right) D^i_\tau d\tau \right] = E_t \left[ \int_t^\infty \left( \frac{m^\tau}{m_t} \right) s^i_\tau C_\tau d\tau \right] \quad (20) $$

in closed form. We then use our expressions for prices to compute returns,

$$ dR_t^i = \frac{dP_t^i + D_t^i dt}{P_t} - r dt $$

and calculate the expected excess returns

$$ E_t [dR_t^i] = -\text{cov} \left( \frac{dm_t}{m_t}, dR_t^i \right) = -\sigma'_m \sigma_R^i, \quad (21) $$

where $\sigma_R^i$ is the diffusion component associated with the returns of asset $i$. Our purpose is to obtain closed form expressions for expressions (20) and (21) and relate them to the parameters in our model.

III.A General Results

III.A.1 The total wealth portfolio

It is useful to start characterizing some basic properties of the total wealth portfolio as the intuition for some of these results becomes useful later.
Proposition 1: The price/consumption ratio, the expected excess return and diffusion terms of the total wealth portfolio are, respectively:

\[
\frac{P_{TW}}{C_t} = \alpha_{TW}^0 (s_t) + \alpha_{TW}^1 (s_t) S_t^\gamma
\]

\[
E_t [dR_{TW}^t] = (\gamma + \alpha (1 - \lambda S_t^\gamma)) \left\{ \frac{S_t^\gamma \alpha (1 - \lambda S_t^\gamma)}{f_{1t}^{TW}(s_t) + S_t^\gamma \sigma_c} + \sum_{j=1}^{n} w_{jt}^{TW} \sigma_{CF,t}^j \right\}
\]

\[
\sigma_{R,t}^{TW} = \frac{S_t^\gamma \alpha (1 - \lambda S_t^\gamma)}{f_{1t}^{TW}(s_t) + S_t^\gamma \sigma_c} + \sum_{j=1}^{n} w_{jt}^{TW} \sigma_{D}^j (s_t)
\]

where the functions \(\alpha_{TW}^0 (s_t), \alpha_{TW}^1 (s_t), f_{1t}^{TW}(s_t)\) and \(\{w_{jt}^{TW}\}\) are given in Appendix.

The results are similar to the ones found by Campbell and Cochrane (1999) and MSV, and we refer to those papers for more detail. Briefly, the price/consumption ratio of the total wealth portfolio is increasing in the surplus consumption ratio \(S_t\). This is intuitive: a high surplus consumption ratio implies a low local curvature of the utility function, a “less risk averse” attitude by part of the representative agent, and this in turn translates into higher price-dividend ratios.

Differently from Campbell and Cochrane (1999) and MSV, the total wealth portfolio depends also on the entire vector of shares \(s_t\). The reason is that in our setting, the general equilibrium restriction (5) generates a mild predictability in consumption growth (see Assumption 1). The functions \(\alpha_{TW}^0 (s_t)\) and \(\alpha_{TW}^1 (s_t)\) are typically decreasing in expected consumption growth, because in our set up the elasticity of intertemporal substitution is less than one. Thus, this component implies that an increase in \(\mu_c (s_t)\) result in lower prices.

Turning to expected returns, the first term in parenthesis reflects the curvature of the utility function of the representative agent, and thus the degree of risk aversion: High curvature parameter \(\gamma\) or low surplus \(S_t\) imply high expected returns. The first term of the expression in brackets is instead linked to discount effects: As shown in the pricing function, changes in \(S_t\) induce a volatility of stock returns which is perfectly correlated with the stochastic discount factor, and thus it is priced. MSV discuss this effect more thoroughly.

The novel term is the second one in the bracket, which is the premium investors require because of changes in expected consumption growth. This second term is typically negative. The reason is that our modeling device induces a mild positive correlation between shocks to consumption growth and shocks to expected consumption growth. Thus, since as explained
earlier, negative shocks to consumption growth are correlated with positive shocks to prices, this component carries a negative premium.

III.A.2 Prices and returns for individual securities

**Proposition 2:** (a) the price of asset \( i \) is given by

\[
\frac{P_i^t}{D_i^t} = \alpha_0^i + \alpha_1^i S_t^\gamma + \alpha_2^i (s_t) S_t^\gamma \left( \frac{\bar{s}_i}{s_i^t} \right) + \alpha_3^i (s_t) S_t^\gamma \left( \frac{\bar{s}_i}{s_i^t} \right)
\]  

(25)

where \( \alpha_0^i, \alpha_1^i \) are positive constants and \( \alpha_2^i (s_t) \) and \( \alpha_3^i (s_t) \) are positive functions of the share vector \( s_t \) given in appendix.

(b) The diffusion term of the return process for asset \( i \) is given by

\[
\sigma_{R_i} = \frac{S_t^\gamma (1 - \lambda S_t^\gamma)}{f_1 (\frac{\bar{s}_i}{s_i^t}, s_t) + S_t^\gamma} \sigma_c + \left( \frac{1}{1 + f_2 (S_t, s_t) \left( \frac{\bar{s}_i}{s_i^t} \right) + \eta_{it}} \right) \sigma_D (s_t) + \sum_{j \neq i} \eta_{jt} \sigma_D (s_t)
\]  

(26)

(c) The expected excess return of asset \( i \) is given by

\[
E_t [dR_i^t] = E_t^{DISC} [dR_i^t] + E_t^{CF} [dR_i^t]
\]

where

\[
E_t^{DISC} [dR_i^t] = (\gamma + \alpha (1 - \lambda S_t^\gamma)) \left( \frac{\alpha S_t^\gamma (1 - \lambda S_t^\gamma)}{f_1 (\frac{\bar{s}_i}{s_i^t}, s_t) + S_t^\gamma} \right) \sigma_c^2
\]  

(27)

\[
E_t^{CF} [dR_i^t] = (\gamma + \alpha (1 - \lambda S_t^\gamma)) \left[ \frac{1}{1 + f_2 (S_t, s_t) \left( \frac{\bar{s}_i}{s_i^t} \right) + \eta_{it}} \right] \sigma_{CF,t} + \sum \eta_{jt} \sigma_{CF,t}
\]  

(28)

with

\[
f_1 (\bar{s}_i / s_i^t, s_t) = \frac{\alpha_0^i + \alpha_2^i (s_t) (\bar{s}_i / s_i^t)}{\alpha_1^i + \alpha_3^i (s_t) (\bar{s}_i / s_i^t)} > 0 \quad \text{and} \quad f_2 (S_t, s_t) = \frac{\alpha_2^i (s_t) + \alpha_3^i (s_t) S_t^\gamma}{\alpha_0^i + \alpha_1^i S_t^\gamma} > 0,
\]

and \( \eta_{jt} \) are given in Appendix.

Proposition 2-(a) shows that the price/dividend ratios of asset \( i \) is increasing with the surplus consumption ratio \( S_t \). As it was true for the total wealth portfolio, in fact, a high \( S_t \) implies a lower risk aversion of the representative agent, and thus higher prices of assets. In addition, however, the price/dividend ratio is increasing in the relative share, \( \bar{s}_i / s_i^t \), which
as shown in expression (11), determines the time varying component of expected dividend growth. A high expected dividend growth results naturally in a higher price/dividend ratio. The last term shows that shocks to expected dividend growth have a different effect depending on the level of the surplus consumption ratio: The higher the surplus consumption ratio the stronger the impact of expected dividend growth shocks on the price-dividend.

Finally, as it was true for the total wealth portfolio, the price of each individual asset also depends on functions of the vectors of shares \( \alpha_2^i (s_t) \) and \( \alpha_3^i (s_t) \). As explained earlier, these functions are typically decreasing in expected consumption growth because the elasticity of intertemporal substitution is less than one in our setting. Thus, an increase in the expected consumption growth decreases all of the assets price/dividend ratios.

As for the expected excess returns the effects are divided in two. These two terms correspond to the two sources of shocks to returns that are shown in the expression for the diffusion component of returns, (26). The first correspond to shocks in the discount factor and the second to shocks to cash-flows, both its own, the second term in expression (26), and shocks to the cash-flows in the rest of the assets in the economy.

**Discount Effects**

The source of this component of the risk premium, \( E_t^{DISC} [dR^i_t] \), is the variation of the aggregate discount — proxied by \( S_t^γ \). To interpret further this term it is useful to notice first that

\[
\frac{\partial P^i_t}{P^i_t} = \frac{S_t^γ}{f_1 (\frac{\bar{s}}{s_t}, s_t) + S_t^γ}.
\]  

(29)

Thus the volatility of an asset’s return is linked to the elasticity of prices to shocks in the variable driving the aggregate discount, which is \( S_t^γ \). The variance of these shocks is linked to

\[
\alpha (1 - \lambda S_t^γ) \sigma_c,
\]

which is the diffusion component of \( dS_t^γ / S_t^γ \), the inverse of our state variable \( G_t \), as it follows from a basic application of Ito’s Lemma to (17). Clearly, only the component of these shocks that covaries with the shocks to the stochastic discount factor is priced which, given (19) is

\[
[\gamma + \alpha (1 - \lambda S_t^γ)] \alpha (1 - \lambda S_t^γ) \sigma_c^2.
\]  

(30)

The component of the asset’s premium that is linked to discount effects is the product of (29) and (30). Cross sectional variation in the discount effects can only be driven by differences in the price elasticity (29), which is in turn driven by the behavior of the function \( f_1 (\bar{s} / s_t, s_t) \).
We have been unable to obtain a general characterization of this function, but for parameter values that are empirically relevant we find that
\[
\frac{\partial f_1(\bar{s}/s^i_t, s_t)}{\partial (\bar{s}/s^i_t)} < 0,
\]
and thus assets with a higher expected dividend growth, as measured by the relative share \(\bar{s}/s^i_t\), display stronger discount effects. The intuition is straightforward: stocks with a high expected dividend growth pay the bulk of its proceeds far in the future. Thus, minor variations in the aggregate discount rate – through the risk aversion of the representative investor – result in large percentage variations of the price of the asset, as also shown in the first term of the diffusion function \(\sigma^i_{R,t}\) in equation (26).

**Cash-flow effects**

The source of premia related to cash-flow shocks, \(E^{CF}_t [dR^i_t]\), has two components to it. The first is related to shocks in the asset’s dividends, which is the second term in expression (26), and the second is related to shocks in the dividends of the rest of the assets in the economy, which, as shown in (25), affect the price of asset \(i\) as well. The logic for the sources of the premia linked to cash-flow shocks is the same as in the discount effects case. First it can be easily shown that the elasticity of the price with respect to shocks to its own dividends is,
\[
\frac{\partial P^i_t/P^i_t}{\partial D^i_t/D^i_t} = \frac{1}{1 + f^i_2(S_t, s_t)} \left(\frac{\bar{s}}{s^i_t}\right) + \eta^i_{it}.
\]
The diffusion term of the dividend process of asset \(i\) is \(\sigma^i_D (s_t)\), and recall that its conditional covariance with consumption growth is denoted by \(\sigma^i_{CF,t}\). The first component of \(E^{CF}_t [dR^i_t]\) is then the component of the dividend shocks that covaries with shocks to the stochastic discount factor multiplied by the effect that these shocks have on the price of asset \(i\), as measured by the elasticity. A similar logic applies to the second term in \(E^{CF}_t [dR^i_t]\). Indeed it can be shown that for \(j \neq i\)
\[
\frac{\partial P^i_t/P^i_t}{\partial D^j_t/D^j_t} = \eta^j_{it}.
\]
As before this component of the premium results from the product of this (cross) elasticity and the priced component of the shock to asset \(j\)’s dividends, \(\sigma^j_{CF,t}\).

How does the current level expected dividend growth, as measured by \(\bar{s}/s^i_t\), affect the cash flow risk component of expected stock returns? Given the conditional covariance of the dividend of asset \(i\) with aggregate consumption, \(\sigma^i_{CF,t}\), the first term of (28) is unambiguous:
Since \( f_z(S_t, s_t) > 0 \), if the asset is “risky”, that is, if \( \sigma_{CF,t} > 0 \), then a high expected dividend growth translates in a lower premium stemming from current dividend volatility. The intuition is also clear: a stock that pays more in the future than today has a relatively low dividend compared to the future. Thus, the risk embedded in current dividends, \( \sigma_{CF,t}^i \), has a relatively low impact on the total risk of stock. In the limit, if the stocks does not pay any dividend today, it cannot have any “cash flow risk”, as there is zero current covariance of dividends with consumption. If instead the asset’s dividends covary negatively with consumption growth (\( \sigma_{CF,t}^i < 0 \)), then a high expected dividend growth increases the risk premium. The argument, of course, is the converse of the previous one.

The effect that the current expected dividend growth of asset \( i \) has on the second term of the cash flow risk component of stock return return (28) is slightly more difficult to interpret. To quantify these effects, the top panel of Figure 1 plots the quantity \( E_t^{CF} [dR_t^i] \) as a function of the unconditional cash flow risk \( \bar{\sigma}_{CF}^i = E [\sigma_{CF,t}^i] \) at the steady state, that is, for the case where \( S_t = \bar{S} \) and \( s_t = \bar{s} \). As it can be seen, the cash flow component of expected return is increasing in \( \bar{\sigma}_{CF}^i \).

It is important to note however, that there is a negative “bias” in this component of expected excess return. Indeed the case \( \bar{\sigma}_{CF}^i = 0 \) still implies a negative expected return from cash flow risk effects. This is due to the second component in (28), which is related to the time variation in the aggregate expected consumption growth. As we discussed in the case of the total wealth portfolio, this component carries typically a negative risk premium. Finally, the bottom panel of Figure 1 plots \( E_t^{CF} [dR_t^i] \) as a function of \( \bar{\sigma}_{CF}^i \) for the case where \( S_t = \bar{S} \) but for a random draw of shares \( s_t \). Although an increasing pattern in \( \sigma_{CF,t}^i \) can be easily seen, cross sectional differences in \( \bar{s}^i/s_t^i \) may make the component \( E_t^{CF} [dR_t^i] \) of an asset with high unconditional cash flow risk \( \bar{\sigma}_{CF}^i \) temporarily lower than that of an asset with lower cash flow risk \( \bar{\sigma}_{CF}^i \).

III.B Two special cases

It is useful to specialize the model and illustrate separately the cash-flow and discount effects that determine premia in the cross section. To do so we restrict ourselves to the version of the habit persistence model studied by MSV.

**Assumption 3:** The preference specification is such that \( \gamma = 1 \).

We consider two polar cases, one where only discount effects are present and there is no cross sectional dispersion in cash-flow risk and a second one where the opposite is true, that
is, there are no discount effects and cross sectional dispersion is exclusively driven by cash-flow risk.

III.B.1 Example I: Discount effects only

**Assumption 4**: All assets have identical cash-flow risk, that is $\theta_{CF}^i = 0$ for all $i = 1, 2, \cdots, n$ so that $\sigma_{CF,t}^i = \sigma_c \sigma_c'$.

Then we can show the following proposition.

**Proposition 3.** Let Assumptions 3 and 4 hold. Then, (a) the price dividend ratio of asset $i$, is given by

$$\frac{P_t^i}{D_t^i} = \Phi^i \left( S_t, \frac{s_t^i}{s_t^i} \right) \equiv a_0 + a_1 \left( \frac{s_t^i}{s_t^i} \right) + a_2 S_t + a_3 \left( \frac{s_t^i}{s_t^i} \right) S_t$$

(31)

where $a_0, a_1, a_2$ and $a_3$ are positive constants given the appendix.

(b) The conditional CAPM holds,

$$E_t \left[ dR_t^i \right] = \beta^i_{DISC} \left( S_t, \frac{s_t^i}{s_t^i} \right) E_t \left[ dR_t^{TW} \right]$$

where $\beta^i_{DISC} \left( S_t, \frac{s_t^i}{s_t^i} \right)$, given in Appendix, is the CAPM beta

$$\beta^i_{DISC} \left( S_t, \frac{s_t^i}{s_t^i} \right) = \frac{\text{cov} \left( dR_t^i, dR_t^{TW} \right)}{\text{var} \left( dR_t^{TW} \right)}.$$  

and $E_t \left[ dR_t^{TW} \right]$ is the expected excess returns of the total wealth portfolio which is given by

$$E_t \left[ dR_t^{TW} \right] = (1 + \alpha (1 - \lambda S_t)) \sigma_{R}^{TW} (S_t) \sigma_c$$

$$\sigma_{R}^{TW} (S_t) = \left[ 1 + \frac{k Y S_t (1 - \lambda S_t) \alpha}{k Y S_t + \rho} \right] \sigma_c.$$  

Equation (31) simply specializes (25) to the present case and the intuition will not be repeated here. Part (b) of the Proposition shows that now, and unlike the case studied in Section III.B, the conditional CAPM holds. To understand why is this the case, it is useful to return to Assumption 1 regarding the consumption growth process. The failure of the conditional CAPM in the general case studied earlier was due to the presence of predictable components in consumption growth, see expression (7). Changes in expected consumption growth, which affect the price and returns of the total wealth portfolio, are not necessarily associated with
changes in the stochastic discount factor, \( m_t \), thus breaking the perfect correlation between the stochastic discount factor and the returns on the total wealth portfolio that is needed in order for the conditional CAPM to hold. Now instead when \( \theta_{CF}^i = 0 \), consumption growth has no predictable components and movements in the return of the total wealth portfolio are exclusively driven by shocks to the stochastic discount factor and the conditional CAPM holds as a result.

The next proposition characterizes the behavior of the conditional beta as a function of \( \bar{\sigma}^i / s_t^i \) and then fully characterizes as well the cross sectional dispersion in expected excess returns.

**Proposition 4:** Let Assumptions 3 and 4 hold. Then, the conditional CAPM beta

\[
\beta^{\text{DISC}}_i (S_t, \bar{\sigma}^i / s_t^i) \text{ is such that}
\]

\[
\frac{\partial \beta^{\text{DISC}}_i (S_t, \bar{\sigma}^i / s_t^i)}{\partial (\bar{\sigma}^i / s_t^i)} > 0,
\]

and, in particular,

\[
\beta^{\text{DISC}}_i (S_t, \bar{\sigma}^i / s_t^i) > (\<)1 \iff \bar{\sigma}^i / s_t^i > (\<)1 \text{ and } \beta^{\text{DISC}}_i (S_t, 1) = 1.
\]

Expression (32) shows that for any value of the surplus consumption ratio the discount beta is increasing in the asset’s relative share \( \bar{\sigma}^i / s_t^i \). The reason is that “high duration” assets deliver dividends in the distant future, and thus their prices are particularly sensitive to changes in the aggregate discount, which is regulated by \( S_t \). These assets are then riskier than otherwise identical assets with lower duration. In particular, and as shown in (33), assets that are currently a lower (higher) fraction of consumption than their long run contribution, have betas that are above (below) one. This can be seen in Figure 2 where we plot \( \beta^{\text{DISC}}_i (S_t, \bar{\sigma}^i / s_t^i) \) as a function of the relative share and the surplus consumption ratio. Notice the cross sectional dispersion in betas is less pronounced when \( S_t \) is high: During booms the aggregate equity premium is low and thus the prices of all assets are mainly driven by the expected dividends in the far future. Mean reversion in dividend growth then implies that the variation in the aggregate discount rate has a similar impact on the prices of assets, and thus all assets will tend to have similar risk. All betas are then close to each other and around 1. In contrast, when \( S_t \) is low and the aggregate discount rate is high, agents discount future dividends by a great deal, and thus the level of current dividend growth matters more. In other words, the
characteristics of whether the asset has high or low duration becomes a key determinant of the riskiness of the asset and, hence the cross sectional dispersion of betas is higher when $S_t$ is low. Notice, finally that in the steady state, that is when $s^i_t = \bar{s}$, betas are equal to one and thus there is no cross sectional dispersion in expected excess returns among this set of assets.

Propositions 3 and 4 have an immediate implication for value versus growth, which we state in the following corollary:

**Corollary 1:** Let Assumptions 3 and 4 hold and consider portfolios of stocks that are sorted by price-dividend ratio. Then, high price-dividend ratio portfolios (growth stocks) yield a higher return on average.

This Corollary has a strong implication: if the relative riskiness of stocks only depended on their sensitivity to changes in the stochastic discount factor, and not risk, then high price-dividend ratio stocks are those that produce high average returns. This implication is in stark contrast with the finding in the data, a signal that the second source of risk — cash flow risk — must take a large part in explaining the existence of a value premium. We now turn to discuss the case of cash flow risk, without discount effects.

**III.C.2 Example II: Cash-flow effects only**

**Assumption 5:** $\alpha = 0$ and $G_t = \bar{G} = \lambda = 1$.

Under Assumption 5 the standard log utility representation with multiple assets obtains. The next proposition characterizes the prices and returns of individual securities in this case. Part (a) is shown in MSV and part (b) in Santos and Veronesi (2005):

**Proposition 5:** Let Assumptions 5 hold. Then (a) The price dividend ratio of asset $i$, is given by

$$\frac{P_i^t}{D_i^t} = \left(\frac{1}{\rho + \phi}\right) + \left(\frac{1}{\rho + \phi}\right) \left(\frac{\phi}{\rho}\right) \bar{s}^i$$

(b) The conditional CAPM with respect to the total wealth portfolio holds and its beta is given by

$$\beta_{CF}^{i}(\bar{s}^i / s_i^t, s_t^i) = \frac{\sigma_{CF, t}^{i} / \sigma_{C}^{2} + \phi \rho \left(\bar{s}^i / s_t^i\right)}{1 + \frac{\phi \rho \left(\bar{s}^i / s_t^i\right)}{\rho \left(\bar{s}^i / s_t^i\right)}}$$

19
Equation (34) shows that, as before, the price-dividend ratio is increasing in $\bar{\sigma}^i/s^i_t$. In addition, part (b) of Proposition 5 also shows that the CAPM beta with respect to the total wealth portfolio holds. The reason why this is true notwithstanding a time-varying expected consumption growth is that under log-utility investors have an elasticity of intertemporal substitution equal to one. As a consequence, the price-consumption ratio of the total wealth portfolio is unaffected by this variation, and it is in fact just equal to $P^T_t/C_t = 1/\rho$. Thus, returns on the total wealth portfolio are perfectly correlated with the stochastic discount factor, and the conditional CAPM representation holds.

The characterization of $\beta^i_{CF}(\bar{\sigma}^i/s^i_t, s_t)$ is straightforward. It is immediate to see that

$$\frac{\partial \beta^i_{CF}(\bar{\sigma}^i/s^i_t, s_t)}{\partial (\bar{\sigma}^i/s^i_t)} < 0 \text{ if and only if } \sigma^i_{CF,t} > \sigma^2_c$$

That is, stocks that are risky (i.e. more risky than consumption itself) have a beta that is decreasing in the expected dividend growth. The reason is that assets with low relative share $\bar{\sigma}^i/s^i_t$ have prices that are mainly determined by the current cash-flows. Naturally then the conditional covariance of cash-flows with consumption growth, given by $\sigma^i_{CF}$, has substantial impact on the riskiness of the asset. This yields a relatively higher risk for assets with low expected dividend growth. This is in stark contrast with the previous case, where, as shown in Proposition 4, it was high expected growth assets the ones that were associated with high betas and thus high expected excess returns. These two examples then illustrate the tension between cash-flow and discount effects. It is the resolution of this tension what determines the cross sectional dispersion in average returns.

What is the implication of cash flow effects for price-dividend sorted portfolios? In this case, the beta of an asset – and thus its expected return – can be high either because the conditional covariance of the asset dividends and consumption growth $\sigma^i_{CF,t}$ is high, or because the expected growth rate of dividends is high, provided $\sigma^i_{CF,t} > 0$. Sorting on price-dividend ratio only picks up the second of the effects in the log utility case, as the price-dividend ratio does not depend on $\sigma^i_{CF,t}$ (see equation (34)). Thus, in general, it is hard to say. However, when $\gamma > 1$, we find that

$$\frac{P^i_t}{D^i_t} = \alpha^i_\theta \times \left[ 1 + \phi \left( \frac{\bar{\sigma}^i}{s^i_t} \right) \frac{s^i_t \alpha_\theta}{1 - \phi \bar{\sigma} \alpha_\theta} \right]$$

In a similar setting, Santos and Veronesi (2005) argue that all of the “risky” assets in the economy should have $\theta^i_{CF} > 0$, while only labor income would have $\theta^i_{CF} < 0$. In this case, sorting by $P/D$ ratio is akin to sorting on betas, and thus they find that high $P/D$ ratio portfolios earn low returns.
where

$$\alpha_i^j = \frac{1}{\rho + \phi + (\gamma - 1)(\pi_c + \theta_i^{CF}) + 1/2\gamma(1 - \gamma)\sigma_c^2}$$

Thus, a high $P/D$ ratio here occurs either if expected dividend growth $\bar{s}/s_i$ is low, or if the unconditional cash flow risk $\theta_i^{CF}$ is low. If the differences in $\theta_i^{CF}$ are substantial, then with $\gamma > 1$ we have that sorting by $P/D$ ratio is akin to sorting on cash flow risk. This has the implication that high $P/D$ sorted portfolios would yield, in average, lower returns.

**IV. SIMULATIONS**

In this section we conduct a simulation study to quantify the discount and cash flow effects discussed in the previous section. In particular, we want to compare the empirical predictions made under the case where only discount effects shape the conditional properties of returns with the case where also cash flow risk effects are present. Importantly, we want to evaluate these trade-offs while making sure that the properties of the market portfolio remain reasonable. This concern imposes some limits on the size of the discount effects.

Table I presents the standard set of stylized facts that constitute the main puzzles in the time series and the cross section. The data set is standard and it is very briefly described in the Notes to Table I. Panel A shows mean and standard deviation for the returns on the market portfolio and the risk free rate. Panel B shows the predictability regressions of Fama and French (1988) and Campbell and Shiller (1988), for two different sample periods, which are meant to emphasize that long horizon return forecastability is sensitive to the particular period under consideration. Panels A and B are the standard concern of the equity premium literature. Panel C shows the value premium and its corresponding puzzle, the failure of the CAPM to generate the large cross sectional dispersion in average returns across book-to-market sorted portfolios. Panel C is the standard concern of the cross sectional literature. The ability of a model to match the stylized patterns in A and B is essentially related to the properties of the stochastic discount factor that the model generates. As it is well known, habit persistence models a la Campbell and Cochrane, such as the present one, are relatively successful in reproducing these patterns in the time series of aggregate returns. Panel C instead is related to both discount and cash-flow effects and it is the focus of our investigation but without loosing sight of the model’s ability to stay close to the facts in Panel A and B.

**IV.A Details of the simulation**

We simulate the model presented in Section II.B. We simulate 10,000 years of quarterly data for 200 firms. Table II contains the parameter values that are going to be used throughout.
These values were chosen to generate moments for our simulated economy that are close to their empirical counterparts in Table I. Panel A of Table II contains our choice of parameters for the consumption process and preferences. Average consumption growth, $\bar{c}$, is set at 2% and the standard deviation is set at 1.5%. This latter value should be measured against a standard deviation of consumption growth for the postwar sample of 1.22% and the one for the longer sample starting in 1889, which is 3.32%. As for the preference parameters, our choice of $\gamma$ is between the values used by MSV, $\gamma = 1$, and the value used by Campbell and Cochrane (1999), $\gamma = 2$. Although the process that is modeled is $G_t = S_t^{-\gamma}$, it is more intuitive and clear to think about our choices in terms of the local curvature of the utility function, $\gamma S_t^{-1}$, as it is the economically relevant magnitude for the effects of interest. Our choices imply a steady state value of the local curvature of the utility function of $\gamma S^{-1} = 48$, higher than the already high value of Campbell and Cochrane (1999) which is 35. The minimum value of this local curvature is 27.75. Finally the parameter $k$ and $\alpha$ are similar to the values chosen by MSV.

As for the share process, we assume that all of the 200 simulated assets have the same steady state contribution to overall consumption, $s = 1/200 = .005$. Also the speed of mean reversion is set at $\phi = .052$, a value that is only slightly lower than the one estimated by MSV for the overall market portfolio. The key parameter in our simulation though is the one that controls differences in cash-flow risk, $\theta_{CF}$. Our general equilibrium setting requires that this parameter is symmetrically distributed around zero, as the aggregate market must have a variance equal to the one of consumption. We assume that $\theta_{CF}$ are distributed between $-.0032$ and $.0032$. Thus for every asset with a cash flow parameter $\theta_{CF}$, there is another one with a cash-flow parameter $-\theta_{CF}$. This parameter implies correlation coefficients between consumption growth and dividend growth in the interval $[-.37, .42]$. To establish some comparison, Campbell and Cochrane (1999) report a correlation between aggregate dividends and consumption of about .20. Clearly the presence of idiosyncratic risk lowers the correlation coefficient of consumption and dividend growth of individual securities. Our paper though is one where we are imposing the general equilibrium restriction that $\frac{P_n}{D_j} = 1$, so that, by construction, the correlation between aggregate dividends and consumption has to be 1.

Finally, throughout we simulate the 200 assets and then we sort these assets into ten portfolios according to their price-dividend ratio in an effort to mimic the standard procedure used in the cross sectional literature. We focus our analysis on these ten portfolios.

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15 See Campbell and Cochrane (1999) Table 2.
IV.B Cash-flow effects, discount effects and the value premium

As emphasized in the previous section, if cash-flow effects are absent, that is, if $\theta_{CF} = 0$, and only discount effects matter, a positive relation obtains between price-dividend ratios and average returns. This can be seen in the top left panel of Figure 3, which plots the average log of the price dividend ratio of the ten sorted portfolios versus the corresponding average excess returns for the parameter values indicated in Table II, with the only exception of $\theta_{CF}$ which is set to zero for all assets. As predicted by theory, discount effects induce a counterfactual relation between the price-dividend ratio and average returns. The data clearly indicates that strong cash-flow effects are needed to generate the value premium.

The top-right panel of Figure 3 shows instead the outcome of the simulation when the value of the parameters are chosen as in Table II, including the value of $\theta_{CF} \in [-.0032, .0032]$. In this case the model generates the empirically observed negative relation between the log of the price dividend ratio and average returns. What is key to obtain this effect is the strength of the cash-flow effects: As we decrease the range of $\theta_{CF}$, we obtain less and less of the negative relation between $P/D$ ratios and average returns.

Whether the discount effects are quantitatively plausible can only be assessed by looking at the model’s ability to reproduce the main properties of the market portfolio. Table III Panel A shows that the model generates a sizable market premium of 4.40%, which is certainly lower than the 7.7% premium observed in the sample, but it is much higher than the equivalent case without habit. In fact, even assuming that the representative agent has power utility with curvature parameter $\gamma^* = \gamma_s^{-1} = 48$, that is, equal to the steady state curvature in our model, the case without habit only yields an equity premium $\gamma^* \times \sigma^2 = \gamma_s^{-1} \times \sigma^2 = 1\%$. Yet, 4.40% average return obtained in our model is below the average in the sample, and we discuss in Section IV.F below the reasons behind the model’s lack of performance in this dimension. Discount effects though are linked to the variation of the stochastic discount factor, which is reflected on the variance of market returns and the model does well on this account: The standard deviation of market returns in 13.60%, which is only slightly lower than its empirical counterpart, which is 16.25%.

The model also generates the pattern of predictability that is observed in the data. When we run the standard predictability regression of long horizon returns on the dividend yield of the market portfolio we obtain coefficients that are of the same order of magnitude as those in the data and that, trivially with a large sample of 10,000 years, are strongly statistically significant. Moreover, and as it is consistent with its empirical counterpart as
well, the coefficients and the $t$-statistics increase with the horizon. Finally, the $R^2$ is similar to the $R^2$ for the sample period 1948-2001, which, recall is very different than that for the 1948-1995 period. The reason, of course, is that the latter does not include the extraordinary market of the 90s and its subsequent correction, a period that saw high realized returns and high prices, the opposite of what the predictability regression prescribes. In summary then, the discount effects seem enough to generate, to a reasonable degree, the empirical properties of the return data for the market portfolios.

As mentioned the only caveat to this is precisely the level of the equity premium which is lower than the average excess return observed in the data. This is reflected as well in the level of the average excess returns of our simulated decile portfolios. Indeed, as already seen in the top-right panel of Figure 3, they display the decreasing relation with prices that is a minimal requirement to approach the data but they are consistently below their empirical counterparts. Still the value premium, the difference between the average excess returns of value stocks, those assets with low prices relative to dividends, and growth stocks, those with high prices, is slightly above 5%, which is very close to the empirical counterpart. The model misses the level slightly but seems to capture cross sectional differences well.

**IV.B.1 The source of the value premium**

In order to gauge the source of the value premium in our model, it is convenient to turn to Figure 4. The first two panels plot the two components of the expected return, i.e. the discount risk component (top panel) and the cash flow risk component (middle panel), against the relative share $\frac{s_i}{S_t}$. The bottom panel plots the total expected return. In each panel, these quantities are plotted for various levels of the asset’s unconditional cash flow risk $\sigma_{CF}^2 = E \left[ \text{cov}(dD_t, dC_t) \right]$. In all cases, the level of surplus $S_t$ is set to its steady state $\overline{S}$. Starting with the top panel, we see that the discount risk component of expected return is increasing in the relative share $\frac{s_i}{S_t}$, i.e. with expected dividend growth. This finding confirms the intuition developed in Section III: Stocks whose cash flows are expected further down in the future are more sensitive to changes in the stochastic discount factor and thus command a higher premium. A second important effect, however, is that the discount risk component of expected returns does depend on the asset’s unconditional cash flow risk $\sigma_{CF}^2$. In other words, stocks with higher cash flow risk $\sigma_{CF}^2$ have a larger discount risk component in expected returns. The intuition is simple: As discussed in Section III, stocks with a higher $\sigma_{CF}^2$ are riskier and as a consequence have lower prices. It follows that changes in the stochastic discount factor have a larger impact, in percentages, on the prices of assets with higher levels
of cash flow risk.

The middle panel of Figure 4 plots the cash flow risk component of expected returns. As shown in Example II in Section III.B.2, we find that for stocks with high cash flow risk, a higher expected dividend growth implies a lower expected return. The reason, recall, is that the current size of dividends compared to the future matters in determining the size of cash flow risk of the stock, as it determines the level of covariation with aggregate consumption. In addition, a higher unconditional cash flow risk $\sigma_{CF}$ translates into a higher expected return. The bottom panel finally reports the total expected return for each asset that is obtained by adding to the discount risk component the cash flow risk component of stock returns.

Turning to the source of the value premium, we notice that in our model sorting by price-dividend ratios is akin to sorting both by cash flow risk $\sigma_{CF}$ and expected dividend growth $\bar{s}/s_i$. In particular, we have that value stocks (assets with low $P/D$ ratios) must have high $\sigma_{CF}$ and low expected dividend growth $\bar{s}/s_i$. In the bottom panel of Figure 4, this combination of low $\bar{s}/s_i$ and high $\sigma_{CF}$ corresponds to the area around the upper left corner of the plot and, thus, as already discussed, have high expected returns. Conversely, growth stocks (assets with high $P/D$ ratios) must have a combination of low risk, that is, low $\sigma_{CF}$, and high expected dividend growth, that is, high $\bar{s}/s_i$. This combination can be found on the lower-right corner of the plot. As it can then be seen, intuitively, value stocks will command a high premium, and growth stocks a low (and even negative) premium, as we find in the simulation results in Table III.

IV.C The CAPM and other asset pricing models

Our model allows us to assess the performance of some of the asset pricing models that have been proposed to address the spread in average returns of “price” sorted portfolios. We present evidence based on time series regressions, contained in Table IV, as well as on cross-sectional regressions, which are in Table V. The reason for presenting evidence pertaining to both is that, as we will see, they illuminate slightly different aspects of the performance of these models. We focus on three standard models: The CAPM of Sharpe (1964) and Lintner (1965), the Fama and French (1993) factor model, and a version of the conditional CAPM model, similar to some of the specific models proposed recently to address the failure of the unconditional CAPM.
IV.C.1 The CAPM

a. The CAPM and the value premium puzzle

Any asset pricing model that generates the value premium has also to be consistent with the value premium puzzle, namely, the inability of the CAPM to price “market” value sorted portfolios. The value premium puzzle can be seen in the last line of Table I Panel C, where the CAPM beta is reported. The beta of the sorted portfolios is flat if not slightly decreasing in the market-to-book, at odds with the strong increasing pattern in average returns. The CAPM produces no cross sectional dispersion in its measure of risk when confronted with substantial variation in average returns. To do this more formally we turn to Table IV Panel A where we report the results of time series regressions of returns on the excess returns on the market portfolio,

\[ R_t^i = \alpha^i + \beta^i R_t^M + \epsilon_t \quad \text{for} \quad i = 1, 2, \ldots, 10. \]

We do this for both empirical (Panel A-1) and simulated data (Panel A-2). The panel shows the intercepts in the time series, \( \alpha \), and its corresponding \( t \)-statistic, \( t(\alpha) \). It also reports the beta on the market portfolio, \( \beta^M \) and its \( t \)-statistic, \( t(\beta^M) \). We have omitted the \( t \)-statistic on the loading for the case of simulated data because, consistently with the data as we shall see, they are all strongly significant (well above 100).

Start with the case of the empirical data. Panel A-1. The intercepts, “alphas” of the CAPM time series regressions are large and statistically significant. Growth stocks have large negative intercepts whereas value stocks have large positive ones. The poor performance of the CAPM can also be seen in line 1 of Panel A in Table V, where we report the standard Fama-MacBeth regression. The coefficient is not statistically significant, enters with the wrong sign and the \( R^2 \) is just 11%.

Turn next to the time series regressions in simulated data, Panel A-2. Unlike the case in the empirical data, the betas cross sectionally correlate positively with average excess returns, an important issue on which more below. Still the cross sectional dispersion in betas is not enough to match the cross sectional dispersion in average returns generated by the model. Indeed the pattern and statistical significance of the intercepts in simulated data is strikingly similar to its counterpart in empirical data. A visual impression of this result can be obtained in the bottom-right panel of Figure 3, which shows the average excess returns for the ten decile simulated portfolios plotted against the CAPM fitted returns.

In summary, the cross sectional dispersion of betas is not enough to generate a large cross
sectional dispersion in CAPM fitted excess returns, at least large enough to match the average excess returns on the simulated decile portfolios. While average returns range between 3.03% for high price-dividend ratio stocks and 8.74% for low price-dividend ratio stocks, the “fitted” returns only range between 3.9% and 5.21% (for the fitted returns see Table III Panel C). That is, the model not only generates the value premium but also the value premium puzzle.

b. Cash-flow effects, discount effects and the value premium puzzle

What is the role of the cash-flow and discount effects in generating these patterns? The role of cash-flow effects here is essential. Indeed, the performance of the unconditional CAPM improves considerably if one shuts down the cross-sectional differences in cash-flow risk, that is, if we set $\theta_{CF} = 0$ in (15), as proposed in Example I in Section III.B.1. This can be seen in the left-bottom panel of Figure 3, where, as before, we plot the CAPM fitted returns versus the average excess returns of our simulated decile portfolios, where the simulation is done with the same parameter values shown in Table II with the only exception of $\theta_{CF}^i$ which is set equal to 0 for all assets. Recall that this case generated a counterfactual positive relation between price-dividend ratios and average returns. As it can be seen in the plot, it also generates a remarkable performance for the CAPM, one at odds with empirical evidence. In our framework, the absence of cash-flow effects results in a good performance of the CAPM, even when, as shown in section III.B.1, it only holds conditionally but not unconditionally.\footnote{As is well known, although conditionally $E_t[dR^i] = \beta_t^i E_t[dR^{mkt}]$, when we condition down we obtain $E[dR^i] = E[\beta_t^i] E[dR^{mkt}] + cov(\beta_t^i, E_t[dR^{mkt}])$. In our model, $\beta_t^i$ and $E_t[dR^{mkt}]$ are correlated, as they both depend on $S_t$. Yet, the bottom-left Panel of Figure 2 shows that this covariance must be small.}

The reason for this is not hard to gauge: When only discount effects matter growth stocks are more sensitive to shocks in the aggregate discount, which now is perfectly correlated with the returns on the total wealth portfolio (so that the conditional CAPM holds.) Thus the “beta” of the growth portfolio is very high and, conversely, that of the value portfolio is low. The CAPM performs well but it assigns high premia to the assets that, in the data, have very low average returns.

c. Cross sectional regressions without constraints on the estimates of the market premium

Returning now to the case where cash-flow effects are present, we saw before that the CAPM betas from simulated data are cross sectionally positively correlated with average excess returns. We saw that the spread is not enough to generate a good fit but, critically, there is indeed a slight alignment between average and fitted returns. It follows that given that the...
CAPM betas are ordered in the right direction, cross sectional regressions that impose no constraints on the level of estimated market premium may immediately induce a good fit as measured for instance by the $R^2$. This can be seen in line 5 of Panel B in Table V, where we run the Fama-MacBeth regression in artificial data: The CAPM produces a good fit with an $R^2$ of 89%. Moreover the market enters significantly and with the right sign. The estimated quarterly market premium though is 3.60%, which corresponds to an annualized value above 14%. This number should be compared to the market premium in our model which is 4.40% (see Panel A of Table III.) Thus the CAPM “works” in our model at the expense of an unreasonable level in the market premium. This message has recently been emphasized by Lewellen and Nagel (2005) and Daniel and Titman (2005): A small but slightly positive cross sectional covariation between betas and average returns can result in the unwarranted support of asset pricing models that fail to impose economically based restrictions on the size of the premia of the proposed factors.

To reiterate the CAPM fails dramatically in our model but a cross sectional regression which is oblivious to the estimate of the market premium is likely to perform well and thus generate misleading support for the CAPM. Clearly this point extends to other models that fail to impose economically based restrictions on the size of the premia of the proposed factors.

IV.C.2 The Fama and French (1993) model

a. Cash-flow effects, discount effects, and HML

The Fama and French (1993) model has become a standard benchmark in asset pricing tests. How well does it work in our set up? To answer this question we construct an HML factor in artificial data. To do so we construct a portfolio that is long the three top decile portfolios and short the bottom three obtained in simulated data and described in Table III Panel C.

To investigate further HML Panel C also reports the average cash-flow risk parameter $\theta_{CF}$ for each of the decile portfolios. Recall that cross sectional differences in the expected covariance between consumption and dividend growth are exclusively driven by this parameter, see expression (15). Notice that there is a clear ordering of average $\theta_{CF}$ across decile portfolios. Value stocks, say portfolio 10, has a much larger value of $\theta_{CF}$ than growth stocks, say portfolio 1. HML then captures cross sectional variation in $\theta_{CF}$ across price-dividend sorted portfolios. Recall that in our framework, as we saw in Section IV.B, the value premium is present only when cash-flow effects are large enough and this is exactly what HML summarizes. Still, it
is important to emphasize that differences in cash flow risk $\theta_{CF}$ also yield differences in the impact that discount effects have on expected returns: Assets with large cash-flow risk have low prices and thus shocks to the aggregate discount have a larger percentage impact on prices. HML then is not a factor capturing cash-flow risk solely, but also, partly, the size of discount risk.

b. Time series and cross sectional regressions evidence

Table IV Panel B presents the results of time series regressions,

$$R_t^i = \alpha_i + \beta_{M}^i R_t^M + \beta_{HML}^i R_t^{HML} + \epsilon_t \quad \text{for} \quad i = 12, \ldots, 10.$$ 

Panel B-1 shows the results in the case of the empirical data. The results are well known. The intercepts go down considerably and only one of them is statistically significant; value (growth) stocks have a large (small) loading on HML and the inclusion of HML in the time series regression collapses the betas on the market portfolio around 1 (see Fama and French (1993, page 21-26)).

Panel B-2 shows the time series regression in simulated data. Again we do not report the $t-$statistic on the loadings on the market and HML as they are all above 100. Turning first to the loadings on the market portfolio, notice that, as it was the case in the empirical sample, adding HML to the time series regressions has the effect of reducing the spread in the estimates of $\beta_{M}^i$: The market beta of growth stocks, portfolio 1, which was .88 in the previous case (see Panel A) increases to .95 and that of the value stocks, portfolio 10, which was 1.19 goes down to .98. As Fama and French (1993) note this pattern is related to the correlation between the market and the returns on HML, which is negative in the data and as we document below, it is also negative in our simulations.

As for the loading on the HML portfolio notice that it has a strong cross sectional variation which reflects the cross sectional variation in the underlying cash-flow risk of the different portfolios. Indeed the loading on HML of the growth portfolio is $-.28$ whereas that of the value portfolio is 1.01. That HML helps considerably in addressing the value premium puzzle in simulated data can be seen in the size of the intercepts of the time series regressions, which drop considerably relative to the size of the intercepts when only the market portfolio is present. Moreover there is no longer any pattern in the variation of the intercept across decile portfolios, which shows that HML is capturing the systematic pattern of misspricing documented in Panel A. Notice that, with the exception of the growth portfolio, the intercepts
are all statistically significant but that again they are considerably lower than when the only factor is the market.

The evidence in the Fama-MacBeth regression confirms the time series evidence. Line 2 of Table V Panel A shows that HML enters significantly and the estimated size of the premium on HML is very close to the average excess return of the HML portfolio. This is also the case in our simulated regression, which is shown in line 6 of Panel B in Table V. The coefficient on the loading on HML is almost identical to its empirical counterpart and, once annualized, close to our estimated average excess return of the HML portfolio, which is 3.91%. The only caveat is that the market portfolio is significant in our simulated Fama-MacBeth regressions whereas it is not in the empirical data. Yet, this table shows that the inclusion of HML in the cross-sectional regression aligns the portfolios correctly, as the intercept is now close to zero (with t-statistics equal to .99 even with 40,000 observations) and the (quarterly) market premium equals 1.35%, which annualized is 5.8%, still higher than the average market return in simulation (4.40%), but much smaller than the 14% obtained for the CAPM case.

**IV.C.3 Conditional asset pricing models**

Conditional asset pricing models have been proposed recently to address the failure of traditional models such as the CAPM to address the cross-sectional variation in average returns in book-to-market sorted portfolios. The idea, as advanced by Hansen and Richard (1987), is that the CAPM may fail unconditionally but may hold conditionally. Thus tests of the CAPM that ignore the use of conditioning information are simply misspecified. Researchers have reacted to this observations by using as a proxy for investors’ information set variables that are known to forecast returns in the time series. Typically these models result in testing a multifactor model where the additional factor, other than the market, is the market itself interacted with the proposed conditioning variables.

Lines 3 and 4 in Panel A of Table V shows that conditioning by the dividend yield of the market portfolio and the cay variable of Lettau and Ludvigson (2001) results also in a coefficient for the instrumented market that is strongly significant. In addition the $R^2$ is an impressive 83% and 81% respectively. Panel B, line 7 shows that our model does also well in this dimension. When we interact the returns of the market portfolio with the simulated dividend yield of the market portfolio we obtain a strongly significant coefficient of similar magnitude to its empirical counterpart.
IV.D The dynamics of the value premium

As already mentioned, the failure of the CAPM has led many researchers to propose variations where the CAPM holds conditionally rather than unconditionally. These researchers parameterize betas as a function of the state variable that forecasts long horizon returns thus capturing changes in investment opportunities. These models go on to show that value stocks are relatively riskier than growth stocks in “bad” times, that is, that the beta of value stocks increase relative to the betas of growth stocks in bad times as defined by the fluctuations of the state variable.17

To ascertain the time series variation of the value premium Table VI Panel A shows the average excess return of the first and tenth decile portfolio as a function of whether the market-to-book ratio of the market portfolio is above or below a certain percentile, denoted by $\bar{c}$. For instance, the first line shows that the average excess rate of return of the first decile (growth) portfolio is 13.18% if the market-to-book of the market portfolio is below the 15th percentile of its empirical distribution and that of the tenth decile (value) portfolio is 23.57%. The value premium is then 10.38%. Instead when the market-to-book is above the 15th percentile the first decile portfolio has an average excess return of 5.73% and the tenth portfolio has one of 10.35% for a total value premium of 4.62%, which is considerably lower than the previous one. This pattern is consistent independently of the percentile that is chosen as a cut-off point. The value premium is higher whenever the market-to-book of the market portfolio is low. Notice that these are also periods where the average excess return of the market is high, as shown in the columns headed by $\bar{R}_M$. In summary then, assuming, as it seems reasonable, that low aggregate prices and high average excess market returns are associated with “bad” times, the value premium is indeed higher in these bad times.

How does the model perform in this dimension? Panel B of Table VI reports the same calculations as in Panel A but with simulated data. The only difference is that, naturally, instead of using the market-to-book we use the price-dividend ratio of the market portfolio to identify the state. The pattern is indeed very similar with the only exception of the level of the premia which is, as already discussed, lower than in the data. The value premium is higher when the price-dividend ratio of the market portfolio is low than when it is high.

17 On the predictability of the returns on HML see Cohen, Polk and Vuolteenaho (2003), who run time series regressions of returns on HML, $R_{HML}$, on the lagged value spread (the difference in log ($BE_t/ME_t$) for a portfolio of high book-to-market firms and low book-to-market firms,) the market $ME/BE$ and other variables. They find evidence for the predictability of the returns on HML (see Table V, page 623.)
It is important then to emphasize that a model, such as the conditional CAPM, that allows for variation in betas that is linked to variation in the market premia, or on some variable that is linked to the price of the market portfolio, is going to automatically improve over the dismal performance of the unconditional CAPM, even when the proposed model is grossly misspecified. This is the common argument in the plethora of conditional CAPM models that have been proposed recently to best the CAPM. It follows that additional restrictions have to be imposed on these models to discriminate among them.

The time series variation of the value premium documented in Table VI also lends some support to Fama and French’s assertion that HML proxies for some unspecified distress factor. The premium for such a factor increases in “bad” times as measured by the low prices of the market portfolio (relative to book in the empirical data and aggregate dividends in the simulations.) Thus assets that have a higher loading on that factor will command a higher premium in bad times. In our framework assets that load on that factor are those that covary relatively more with consumption growth, see the last line in Panel C of Table III that shows the average $\theta_{C,F}^i$ of the different decile portfolios, or assets that have poor expected dividend growth prospects and thus have low price-dividend ratios. Notice that assets in the tenth decile portfolio have on average higher cash-flow risk and thus lower prices relative to dividends and thus are more likely to be value assets.

IV.E Characteristics versus covariances

An important issue in the empirical asset pricing literature is whether the cross sectional dispersion in the average returns on market-to-book sorted portfolios is compensation for risk, as Fama and French (1993) would have it, or is linked to the asset’s characteristic, as argued by Daniel and Titman (1997). To test this proposition these authors propose to form portfolios based on the characteristic, say market-to-book, and the loading on the proposed factor, say HML. If returns are generated by the alternative characteristic model then high factor loading portfolios should have negative intercepts whereas low factor loading portfolios should have positive ones. This is exactly what Daniel and Titman (1997) find and this leads them to conclude that the cross section of average return appears driven by characteristics rather than by loadings on the HML factor.

Ours is a rational asset pricing model and characteristics are not driving the cross section of average returns but, it is interesting to ask whether when taking a misspecified model such as Fama and French (1993), characteristics may appear to drive the cross section of average returns. That is, how do the Daniel-Titman tests behave in our simulated sample?
To answer this question we follow the Daniel-Titman procedure of forming portfolios based on preformation loadings on HML, which stand as a proxy for the future loading, and the price-dividend ratio. Briefly we form three portfolios based on 33% and 66% cut-off points for the price dividend ratio that then we intersect with three portfolios formed on the basis of underlying assets’ loadings on HML. This procedure results in nine portfolios. Portfolio, say MI, is the result of the intersection of the medium price-dividend portfolio (M) and the low HML loading portfolio (l). The results are in Table VII which contains the average excess returns of each portfolio, $\bar{R}$, the average price-dividend ratio of the portfolio, $P/D$, and the intercept associated with the time series regression of the portfolios excess returns on the market excess returns and HML, $\alpha$, and its $t$–statistic, $t(\alpha)$. Finally we report the the loadings on the market and HML calculated using the full sample, $\beta_M$ and $\beta_{HML}$ respectively.

First notice that indeed pre- and post-formation HML loadings align well, which confirms that the preformation loading is a good proxy for future expected loading. Also intersecting price-dividend sorted portfolios with preformation HML loading sorted portfolios does not pick additional variation in the price-dividend ratio except in the assets with very high price-dividend ratios (H), where the sort on preformation loading is a essentially a refined $P/D$ sort.

The table shows that there is considerable variation in average excess returns even when controlling for the HML loading. Take for instance the portfolio with low HML loading (portfolio l): Low price dividend ratio portfolios (L) command a higher premium (5.90%) than medium and high (M and H) price-dividend sorted portfolios (3.95% and 2.68%, respectively.) The pattern is identical for portfolio m and h. This cross sectional variation is indeed associated with cross sectional variation in the full sample loadings, $\beta_{HML}$. Indeed, as it was the case in Daniel and Titman (1997, Table V), the intersection of, say, portfolio l with the price-dividend sort captures substantial variation in the post-formation loading. Indeed, the post-formation loadings are such that $\beta_{HML}^L > \beta_{HML}^M > \beta_{HML}^H$.

The important point in Table VII though are the intercepts. Under the null that the Fama-French model captures the cross section of average returns, the “alphas” should be equal to zero. As we saw in section IV.C.2, they are not in the case of the decile portfolios sorted simply on $P/D$ and they are not equal to zero in this case either. The key point though is the sign of these intercepts. It is indeed the case that low loading portfolios, (Ll, Ml, and Hl) have positive “alphas” with the exception of Hh, which has a negative intercept (though surprisingly it is not statistically significant.) But only one of the three high load portfolios
(Lh, Mh, and Hh) has a negative intercept, portfolio Lh, and it is the only one that is not statistically significant even with 10,000 years of quarterly data. These results hold even when we do a finer double sort and intersect four price-dividend ratio sorted portfolios with five HML loadings sorted portfolios. These results hold even when we do a finer double sort and intersect four price-dividend ratio sorted portfolios with five HML loadings sorted portfolios. High loading portfolios have mostly positive alphas (four out of five), at odds with the predictions of the characteristic model.

IV.F Discussion

IV.F.1 Do value stocks have larger cash-flow risk?

Many of our results depend on the extent to which value stocks are also those with high cash-flow risk. Is this the case? Recently there has been a flurry of research that is remarkably consistent on giving a positive answer to this question. For instance, Campbell and Vuolteenaho (2005) find that the value stocks are the ones with a high “cash-flow beta” and they show that the premium associated with this loading is relatively higher than the premium associated with the “discount beta.” Cohen, Polk and Vuolteenaho (2003) obtain cash-flow betas by regressing firm’s profitability on the market’s profitability and find that indeed value stocks have higher regression coefficients. Bansal, Dittmar, and Lundblad (2005) regress dividend growth on a moving average of consumption growth rates, an approach similar to the one of Parker and Julliard (2005), and find that indeed cash-flow betas are larger for value sorted portfolios (see their Table 1.) Recently as well, and following the previous authors, Hansen, Heaton, and Li (2004) have focused on the valuation of cash-flows that are cointegrated with consumption,

\[ d_t = \lambda c_t + \zeta t + \phi(x_t), \]

where \( x_t \) is a variable driving changes in expected consumption growth. Here an increase in the value of \( \lambda \) also increases the long run covariation of consumption growth. They find that the low (high) BE/ME portfolios have indeed low (high) values of \( \lambda \). That is, value stocks have larger (long run) covariation with consumption growth.

IV.F.2 Is consumption growth predictable by the stock market?

A feature of our model that is different from Campbell and Cochrane (1999) and MSV is the presence of predictable components in consumption growth but as Campbell (2002)

\[ \text{These results are not reported here for reasons of space.} \]

\[ \text{These authors emphasize that whether the parameter } \lambda \text{ is well identified depends on the presence of the time trend. In their absence } \lambda \text{ is only very poorly identified. As they point out, the economic interpretation of these time trends is difficult.} \]
emphasizes, real US consumption growth is not well forecast by the stock market and it is only modestly autocorrelated. It is reasonable then to ask then whether Assumption 1 regarding the drift of the consumption growth process induces a counterfactual predictability in consumption growth. Table VIII shows the results of long horizon predictability regressions of consumption growth on the dividend yield of the market portfolio. The coefficient is statistically significant, which is not surprising given that we are using 10,000 years of quarterly data. Importantly though the dividend yield captures a tiny .6% of the variation of consumption growth at the two year horizon and even less for other horizons. The stock market then captures very little of the predictable components in consumption growth.

IV.F.3 Predictable components in consumption growth and the level of market premia

Notwithstanding the lack of forecastibility of consumption growth, the presence of these predictable components have a strong effect on prices. Indeed, shocks in consumption growth are slightly positive correlated with expected consumption growth. The implication is that negative shocks to consumption growth are associated with negative shocks to expected consumption growth, which in turn tend to increase the price-consumption ratio of the total wealth portfolio. The reason is that our representative consumer has a very low intertemporal elasticity of substitution and thus very strong preferences for intertemporal consumption smoothing. This second effect has the impact of reducing the covariation between consumption shocks and returns, and thus the equity premium. The size of this counterbalancing effect on the equity premium depends on the “size” of the term $\mu_{c,1}(s_t) = s_t^t \theta_{CF}$, see Assumption 1. If all $\theta_{CF}$ are close to zero, then this effect is negligible, but if they are large – as it is necessary to obtain substantial cash flow effects – then the intertemporal substitution effect will be large. This is the key to the lower market premium in our simulations when compared to Campbell and Cochrane (1999) or MSV. Notice that the intertemporal substitution mechanism also impacts the predictability regression: A negative shock on consumption growth does increase the premium but does not lower the price as much because, on average, negative consumption growth shocks are associated with negative shocks to expected consumption growth. Market prices drop by less than in the absence of changes in expected consumption growth and this weakens the ability of price-dividend ratios to forecast long horizon returns.

IV.F.4 On the definition of firms

Our model is very stylized with respect to the definition of firms. In our setting, a firm is infinitely lived with a cash flow risk parameter that is chosen at time “0” and kept constant.
throughout. Both assumptions are unrealistic: Firms do not live forever and, importantly, it is likely that the riskiness of a firm cash-flows change with the life cycle of the firm. For instance, Fama and French (2004) report that new lists have cash flow characteristics that are quite different from old firms. Similarly, Pastor and Veronesi (2003) show that young firms have higher $ME/BE$ ratios than old firms, everything else equal. Although it would be valuable to incorporate such assumptions in our model, we do not believe the conclusions would change. The reason is that our model’s predictions are about “price sorted” portfolios and their characteristics depend, to a first degree, on expected dividend growth and the level of cash flow risk. For instance, suppose we assume that every period new firms are born, and, consistently with a life cycle model, new firms have higher expected dividend growth (they are born with a low relative share) and, say, a low cash flow risk parameter $\theta_{CF}$. Assume also that during the life cycle, $\theta_{CF}$ increases, as fixed costs and transformation costs increase, thereby increasing the sensitivity of cash flows to the business cycle. In equilibrium, when there are many assets, we will still have a cross-section of stocks that are characterized by either high or low cash flow risk, as well as either high and low expected dividend growth. The sorting procedure would then pick the same type of stocks that the current sorting procedure does. That is, growth stocks would be those with high expected dividend growth and low cash flow risk, while value stocks would be those with low expected dividend growth and high cash flow risk. The implications for returns would then be the same as in the current setting.

V. CONCLUSIONS

Two effects combine to determine the cross section of average returns: discount and cash-flow effects. We have shown that if only the former are present then assets with high valuation ratios as measured by the price-dividend ratio, growth stocks, command a high premium and assets with low price-dividend ratios command a low premium. The reason is that if only discount effects are present, then an asset’s conditional premium only depends on how sensitive is its price to shocks in the stochastic discount factor. This sensitivity is directly linked to the asset’s duration, that is, it is linked to whether the asset is delivering the bulk of its contribution to consumption far in the future or not. Assets with longer duration are more sensitive to shocks in the stochastic discount factor than otherwise identical assets with lower duration. These assets are also those with stronger dividend growth and, as a consequence, have higher price-dividend ratios. The result is then a positive relation between price dividend ratios and average excess returns.
This last implication is counterfactual. The value premium is the observation that “value” stocks, stocks with low prices to fundamentals have, on average, higher excess returns than “growth” stocks. This implication can only be obtained in the presence of strong cash-flow effects. The intuition is simple enough: assets with high cash-flow risk have lower prices and higher expected excess returns in order to compensate investors for the positive covariation between the assets’ cash-flow and consumption growth. Thus cash-flow effects are the key ingredient to deliver the value premium.

The magnitudes of these cash-flow effects can only be assessed in the presence of realistic discount effects. We know that strong discount effects are needed to deliver plausible quantitative implications for the return properties of the market portfolio. We have proposed a framework, which is a version of the model used by Campbell and Cochrane (1999) and Menzly, Santos, and Veronesi (2004), where the stochastic discount effects are such as to deliver plausible implications for the market portfolio. A key ingredient is that the representative investor’s attitudes towards risk fluctuate depending on the realization of shocks to aggregate consumption. We have shown that this effect results in making value stocks relative riskier than growth stocks in downturns, as measured by negative shocks to aggregate consumption. This is natural: Agents demand a relatively higher compensation for holding assets with cash-flows that covary positively with consumption growth when faced with adverse shocks. Thus cash-flow effects have a conditional effect as well. To put it differently, cross sectional dispersion in cash-flow risk results not only on cross sectional dispersion in unconditional average excess returns, but also on variation in the conditional premia that further reinforces the unconditional effect. Indeed we show that the value premium increases considerably during bad times. This may explain why the two different strands of rational explanations proposed to address the value premium, conditional asset pricing models such as Lettau and Ludvigson (2001) and models based on cash-flow risk, such as Parker and Julliard (2005) and Bansal Dittmar and Lundblad (2005), have been relatively successful: They are both sides of the same coin in the presence of discount effects.

We have used the model to interpret the factor model of Fama and French (1993). In particular we have constructed an HML factor and found that it captures cross sectional dispersion in cash-flow risk. Assets with a high cash-flow risk have, mechanically, a high loading on HML. The premium on HML varies over time, as already mentioned, so the Fama and French model performs well because it captures the sources of unconditional cross sectional variation in average excess returns through the load on HML as well as conditional variation.
through the fluctuations in the premium on HML.

The model then provides a coherent view of the time series and the cross section of average returns but it is to the latter that the present paper makes a contribution by quantitatively assessing what is needed to deliver the value premium. Much remains to be done, of course. In particular it would be interesting to investigate the small sample properties of our model to address one question where our paper is slightly at odds with the empirical results, namely, whether a rational asset pricing model such as Fama and French (1993) can be distinguished in finite sample from a characteristic model. That is, to what extent can these cash-flow effects that so key a role play in the cross section be uncovered with the sample sizes at our disposal?
REFERENCES


Lettau, Martin and Jessica Wachter (2005), “Why is long-horizon equity less risky? A duration-based explanation of the value premium” *manuscript*, NYU, Stern School of Business.


APPENDIX

The habit dynamics: If \( X_t = \lambda \int_t^t e^{-\lambda(t-s)} C_r d\tau \) we have \( dX_t = \lambda (C_t - X_t) dt \). Define then \( G_t = f(C_t, X_t) = (C_t / (C_t - X_t))^\gamma \). We then have

\[
\begin{align*}
f_C &= -\gamma G_t \left( G_t^\frac{1}{\gamma} - 1 \right) C_t^{-1} \\
f_{CC} &= \left\{ \gamma (\gamma - 1) G_t \left( G_t^\frac{1}{\gamma} - 1 \right)^2 + 2\gamma \left( G_t^\frac{1}{\gamma} - 1 \right) G_t^{\frac{1}{\gamma} + 1} \right\} C_t^{-2} \\
f_X &= \gamma G_t \left( \frac{1}{(C_t - X_t)} \right)
\end{align*}
\]

where we used \( G_t^{\frac{1}{\gamma}} = C_t / (C_t - X_t) \) and \( G_t^{\frac{1}{\gamma}} - 1 = X_t / (C_t - X_t) \). Ito’s Lemma then yields

\[
dG_t = \left\{ \mu_C (G_t) - \sigma_C (G_t) \mu_{\{c,1\}} (s_t) \right\} dt - \sigma_C (G_t) \sigma_c dB_t
\]

where

\[
\begin{align*}
\mu_C (G_t) &= \gamma \lambda G_t + \frac{1}{2} \gamma (\gamma - 1) G_t \left( G_t^{\frac{1}{\gamma}} - 1 \right)^2 \sigma_e^2 + \gamma \left( G_t^{\frac{1}{\gamma}} - 1 \right) G_t^{\frac{1}{\gamma} + 1} \sigma_e^2 - \sigma_G (G_t) \mu_e \\
\sigma_C (G_t) &= \gamma G_t \left( G_t^{\frac{1}{\gamma}} - 1 \right)
\end{align*}
\]

Proof of Proposition 1. This is a corollary to Proposition 2, and it is proved below.

Proof of Proposition 2. Part (a). Pricing Formula. The pricing formula is

\[
P_t = E_t \left[ \int_t^\infty e^{-\rho(t-s)} \frac{u_c (C_s, X_s)}{u_c (C_t, X_t)} D_s \, ds \right] = C_t^{\gamma} G_t^{\frac{1}{\gamma} - 1} E_t \left[ \int_t^\infty e^{-\rho(t-s)} C_s^{1-\gamma} G_r s \, ds \right]
\]

We divide the proof in two parts: First, we obtain a general pricing formula which depends on the state variables. Second, we obtain analytical solutions for the coefficients of these state variables.

Part a.1: A pricing formula. For this proof, it is convenient to rewrite the share processes in its general form as

\[
ds_t = \sum_{i=1}^n s_t^i \lambda_{ji} dt + s_t^i \left( \nu^i - s_t^i \nu \right) dB_t
\]

where \( \lambda_{ji} = \phi \sigma_f \) for \( i \neq j \), and \( \lambda_{ii} = -\sum_{j \neq i} \lambda_{ij} = -\phi \sum_{j \neq i} s_j = -\phi (1 - \sigma_f) = \phi \sigma_f - \phi \). Define the two quantities

\[
q_t^i = C_t^{1-\gamma} G_t s_t^i \quad \text{and} \quad p_t^i = C_t^{1-\gamma} s_t^i
\]

and the \( 2n \times 1 \) vector \( y_t = [q_t, p_t] \). An application of Ito’s Lemma and tedious algebra shows

\[
dy_t = \hat{\Lambda} y_t dt + \Sigma_{y,t} dB_t
\]

where

\[
\hat{\Lambda}_y = \begin{bmatrix} \Lambda' + \hat{\Theta}_q & \hat{\Theta}_q \\ 0 & \Lambda' + \hat{\Theta}_p \end{bmatrix}
\]
\( \Lambda = \phi (\mathbf{F} \times \mathbf{1}_n), \) \( \Theta_i \) for \( i = q, p, qp \) are diagonal matrices with \( ii \) element given by

\[
\begin{align*}
\hat{\theta}_i &= (1 - \gamma) \pi_i - \frac{1}{2} \gamma (1 - \gamma) \sigma^2_k - k - (1 - \gamma) \sigma^2_\alpha + (1 - \gamma) \theta_i - \alpha \theta^i \\
\hat{\theta}_{qp} &= k \mathcal{G} + (1 - \gamma) \sigma^2_\alpha \lambda + \alpha \lambda \theta^i \\
\hat{\theta}_p &= (1 - \gamma) \pi_i - \frac{1}{2} \gamma (1 - \gamma) \sigma^2_k + (1 - \gamma) \theta_i
\end{align*}
\]

and \( \Sigma_{\hat{\theta},t} \) is an appropriate matrix. Assuming existence of the expectation in the pricing function, we can apply Fubini’s theorem

\[
P_t = C_t^\gamma G_t^{-1} E_t \left[ \int_t^\infty e^{-\rho(t-\tau)} y^t d\tau \right] = C_t^\gamma G_t^{-1} \int_t^\infty E_t \left[ e^{-\rho(t-\tau)} y^t \right] d\tau
\]

The expectation in the integral can be computed as follows: Let \( \omega \) be the vector of eigenvalues of \( \hat{A}_y, \) \( \left[ \omega^{(r-t)} \right] \) the diagonal matrix with \( ii \) element given by \( \omega^{(r-t)} \) and \( U \) the matrix of associated eigenvectors. Then, we can write

\[
E_t \left[ e^{-\rho(t-\tau)} y^t \right] = \mathbf{U} \cdot \left[ e^{\omega^{(r-t)}} \right] \cdot \mathbf{U}^{-1} \cdot y^t e^{-\rho(t-\tau)} = \sum_{k=1}^{2n} \sum_{j=1}^{2n} u_{ik} e^{(\omega_k - \rho)(t-\tau)} \left[ u_{jk}^{-1} \right] y_{jt}
\]

where \( \left[ u_{jk}^{-1} \right] \) is the \( jk \) element of \( U^{-1} \). Substituting into the expectation, and taking the integral, we find

\[
\int_t^\infty E_t \left[ e^{-\rho(t-\tau)} y^t \right] d\tau = \sum_{k=1}^{2n} \sum_{j=1}^{2n} u_{ik} \frac{u_{jk}^{-1}}{\rho - \omega_k} y_{jt} = \sum_{j=1}^{2n} b^j y_{jt}
\]

where

\[
b^j = \sum_{k=1}^{2n} \frac{u_{ik} u_{jk}^{-1}}{\rho - \omega_k}
\]

Below, we obtain these coefficients in closed form. Note, however, that by substituting \( y_{jt} = q_{jt} \) for \( j = 1, ..., n \) and \( y_{jt} = p_{j-n,t} \) for \( j = n + 1, ..., n \) we obtain

\[
P_t = C_t^\gamma G_t^{-1} E_t \left[ \int_t^\infty e^{-\rho(t-\tau)} y^t d\tau \right] = C_t^\gamma G_t^{-1} \left( \sum_{j=1}^{n} b^j q_{jt} + \sum_{j=1}^{n} b^j p_{j,t} \right)
\]

\[
= C_t^\gamma G_t^{-1} \left[ C_t^{1-\gamma} G_t \sum_{j=1}^{n} b^j q^j_i + C_t^{1-\gamma} \sum_{j=1}^{n} b^j q^j_i \right]
\]

\[
= C_t \sum_{j=1}^{n} \left( b^j q^j_i + b^j q^j_i \right) q^j_i
\]

**Part a.2: Analytical formulas for** \( b^j_{1,j} \) **and** \( b^j_{2,j} **.** \) We finally obtain a closed form formula for \( b^j \)'s, and thus, of \( b^j_{1,j} \) and \( b^j_{2,j} **.** \) First, note that we can write

\[
b^j_i = \mathbf{U} \cdot \left( \Omega^{-1} \right) \cdot \mathbf{U}^{-1} \mathbf{e}_j
\]

where \( \Omega \) is the matrix with the eigenvalues of \( \mathbf{I}_\rho - \hat{A}_y \) on the principal diagonal. But then, since \( \mathbf{U} \cdot \left( \Omega^{-1} \right) \cdot \mathbf{U}^{-1} = \left( \mathbf{I}_\rho - \hat{A}_y \right)^{-1} \) we have that for \( i = 1, ..., n \) and \( j = 1, ..., 2n \)

\[
b^j_i = \mathbf{U}_i \cdot \left( \mathbf{I}_\rho - \hat{A}_y \right)^{-1} \cdot \mathbf{e}_j
\]

We now explicitly compute these quantities. Define \( B = \left( \mathbf{I}_\rho - \hat{A}_y \right)^{-1} \), so that

\[
B \left( \mathbf{I}_\rho - \hat{A}_y \right) = I
\]
Making this explicit, for every \( i = 1, \ldots, n \) (row) we have
\[
\sum_{j=1}^{2n} b_j^i \left( \mathbf{I}_\rho - \tilde{\mathbf{A}}_y \right)_j = \epsilon_i
\]
where \( \left( \mathbf{I}_\rho - \tilde{\mathbf{A}}_y \right)_j \) is the \( j \)th row of \( \left( \mathbf{I}_\rho - \tilde{\mathbf{A}}_y \right) \) and \( \epsilon_i \) is a \((1 \times 2n)\) row vector with 1 in \( i \)th position, and zero elsewhere. For every \( i \), we have a system of equations that pins down \( b_j^i \) for all \( j = 1, \ldots, 2n \). We now solve this system of equations. To limit the number of indices involved, we do this exercise for \( i = 1 \). Of course, the methodology works for every \( i \). For \( i = 1 \) we have then the following two systems of equations. The first holds for \( j = 1, \ldots, n \) and the second for the remaining \( n \) rows:
\[
\begin{align*}
\phantom{-b_1^i \phi \sigma^1 + b_2^i \phi \sigma^2 + \cdots} & \quad -b_1^i \phi \sigma^1 + b_2^i \phi \sigma^2 - \sum_{j=3}^{n} b_j^i \phi \sigma_j = 0 \quad \text{(row 2)} \\
& \vdots \\
\phantom{-b_1^i \phi \sigma^1 + b_2^i \phi \sigma^2 + \cdots} & \quad -b_1^i \phi \sigma^1 + b_2^i \phi \sigma^2 - \sum_{j=3}^{n} b_j^i \phi \sigma_j = 0 \quad \text{(row j)}
\end{align*}
\]
\[
\begin{align*}
\phantom{-b_1^i \phi \sigma^1 + b_2^i \phi \sigma^2 + \cdots} & \quad -b_1^i \phi \sigma^1 + b_2^i \phi \sigma^2 - \sum_{j=3}^{n} b_j^i \phi \sigma_j = 0 \quad \text{(row 2n)}
\end{align*}
\]
The first set of equation is readily solved. In fact, we can write
\[
\begin{align*}
b_1^1 &= \alpha_q^1 \phi + \sum_{j=1}^{n} b_j^1 \psi^j \\
b_k^1 &= \alpha_q^1 \phi + \sum_{j=1}^{n} b_j^1 \psi^j \quad \text{for} \quad k = 2, \ldots, n
\end{align*}
\]
where
\[
\alpha_q^1 = \frac{1}{\phi + \theta_q^1 - \rho}
\]
Multiply both sides of each row \( k = 1, \ldots, n \) by \( \psi^j \) and sum across rows to obtain
\[
\sum_{j=1}^{n} b_j^1 \psi^j = \sum_{j=1}^{n} \psi^j \alpha_q^1 \phi + \sum_{j=1}^{n} \psi^j \alpha_q^1 \phi \sum_{j=1}^{n} b_j^1 \psi^j
\]
Define the constants
\[
H_q = \sum_{j=1}^{n} \psi^j \alpha_q^1 \phi \quad \text{and} \quad K_q = \frac{1}{1 - \phi H_q}
\]
Solving for $\sum_{j=1}^{n} b_j^1 s^j$ we obtain the quantity

$$\sum_{j=1}^{n} b_j^1 s^j = \alpha_q^1 K_q$$

Thus

$$b_1^1 = \alpha_q^1 + \alpha_q^1 \times \phi s^1 \alpha_q^1 K_q \quad (38)$$

$$b_k^1 = \alpha_q^k \times \phi s^1 \alpha_q^1 K_q \quad \text{for} \quad k = 2, \ldots, n \quad (39)$$

Hence, the first term in the P/C ratio obtained earlier, i.e.

$$\frac{P_1^1}{C_t} = \sum_{j=1}^{n} b_j^1 s^j + \sum_{j=1}^{n} b_{2j}^1 s^j s_i$$

is given by

$$\sum_{j=1}^{n} b_j^1 s^j = \alpha_q^1 s^j + \phi s^j \alpha_q^1 K_q \sum_{j=1}^{n} \alpha_q^k s^j$$

where recall that for $j = 1, \ldots, n$ we defined earlier $b_1^1 = \beta_1^1$.

We now turn to the second system of equations, which for $k = 1, \ldots, n$ can be rewritten as

$$b_{n+k}^1 = \alpha_p^k \phi \sum_{j=1}^{n} b_{n+j}^1 s^j + \beta_1^k \alpha_p^1 \theta_{qp}^p$$

with

$$\alpha_p^k = \frac{1}{(p + \phi - \theta_p^k)}$$

and $b_1^1$ given in (38) - (39). Substitute $b_1^1$ first, to obtain

$$b_{n+k}^1 = \alpha_p^k \phi \sum_{j=1}^{n} b_{n+j}^1 s^j + \alpha_p^1 + \alpha_p^1 \times \phi s^1 \alpha_q^1 K_q \alpha_q^k$$

$$b_{n+k}^1 = \alpha_p^k \phi \sum_{j=1}^{n} b_{n+j}^1 s^j + \alpha_p^1 \phi \alpha_q^1 K_q$$

where

$$\alpha_p^k = \alpha_p^1 \theta_{qp}^p$$

As before, for $k = 1, \ldots, n$ multiply both sides by $\sum \phi s^j$ and sum across $k$’s to obtain

$$\sum_{k=1}^{n} \phi s^k b_{n+k}^1 = \alpha_p^1 \phi s^1 + \left( \sum_{k=1}^{n} \phi s^k \alpha_p^1 \right) \phi \sum_{j=1}^{n} b_{n+j}^1 s^j + \left( \sum_{k=1}^{n} \phi \alpha_p^1 \right) \phi s^1 \alpha_q^1 K_q$$

Let

$$H_p = \left( \sum_{k=1}^{n} \phi s^k \alpha_p^1 \right)$$

and solve for $\sum_{k=1}^{n} \phi s^k b_{n+k}^1$ to find

$$\sum_{k=1}^{n} \phi s^k b_{n+k}^1 = \alpha_p^1 \phi s^1 K_p + \left( \sum_{k=1}^{n} \phi s^k \alpha_p^1 \right) \phi s^1 \alpha_q^1 K_q K_p$$

where

$$K_p = \frac{1}{(1 - \phi H_p)}$$
Substitute back into $b_{n+1}$ and $b_{n+k}$ and find
\[
\begin{align*}
  b_{n+1}^i &= \alpha_{pq}^i + \pi^i g^1
  \\
  b_{n+k}^i &= \pi^i g^k
\end{align*}
\]
where for $k = 1, \ldots, n$
\[
g^1_k = \alpha^1_q \phi \left\{ \alpha_p^k \left( \alpha_{pq}^i \theta_{pq}^i K_p + \left( \sum_{j=1}^{n} \pi^j \alpha_{pq}^j \right) \phi K_p K_p \right) + \alpha_{pq}^k K_p \right\}
\]

Thus, the second part in the price-consumption ratio is given by
\[
\sum_{j=1}^{n} b_{2j}^i s^j = \alpha_{pq}^i s^j + \pi^i \sum_{k=1}^{n} g^k s^i
\]

Generalizing the above derivations for every $i = 1, \ldots, n$, we can finally write
\[
P_i^t = \alpha_0^i + \alpha_1^i \left( s^i \right) + \alpha_2^i \left( s^i \right) S^i
\]
where
\[
\begin{align*}
  \alpha_0^i &= \frac{1}{\rho + \phi - \bar{\theta}} \\
  \alpha_1^i &= \frac{\alpha_{pq}^i}{\rho + \phi - \bar{\theta}^i} \\
  \alpha_2^i (s^i) &= \phi \alpha_{pq}^i (s^i) \alpha_i \\
  \alpha_3^i (s^i) &= \delta_i \sigma^i
\end{align*}
\]
and
\[
\begin{align*}
  \frac{\bar{\theta}^i}{\phi} &= (1 - \gamma) \pi^i - (1 - \gamma) \left( \frac{1}{2} \gamma + \alpha \right) \sigma^2 \alpha + (1 - \gamma - \alpha) \theta^i \\
  \bar{\theta}_{pq}^i &= k \bar{\theta} + (1 - \gamma) \sigma^2 \alpha \lambda + \alpha \lambda \theta^i \\
  \bar{\theta}_{pq}^i &= (1 - \gamma) \pi^i - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 \alpha + (1 - \gamma) \theta^i
\end{align*}
\]

**Part (b) of Proposition 2: The Diffusion Component of Stock Returns**

As shown, we can write
\[
P_i^t = C_t \left( \alpha_0^i s^i + \alpha_2^i (s^i) \pi^i + \left( \alpha_1^i s^i + \alpha_3^i (s^i) \pi^i \right) S^i \right)
\]
Define by $S_t = S^i \alpha (1 - \lambda S^i) \sigma$. Using Ito’s Lemma, it is immediate to see that the diffusion of $dS$ is given by
\[
\sigma_S (S^i) = S^i \alpha (1 - \lambda S^i) \sigma
\]

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Thus, an application of Ito’s Lemma shows that the diffusion term of $P^t_i$ is given by

$$\sigma^s_{R,t} = \sigma_e + \left( \frac{\alpha_0 s_t^i + \alpha_3 s_t^i }{\alpha_0 s_t^i + \alpha_3 s_t^i } \right) S_t^0 \alpha \left( 1 - \lambda S_t^0 \right) \sigma_e + \sum_{k=1}^{n} \left( \frac{\alpha_0 + \alpha_1 s_t^i }{\alpha_0 s_t^i + \alpha_3 s_t^i } \right) 1_{(k=i)} \alpha \phi q K_t^i \alpha_k^i + g_k^i \sigma_k^i \left( s_t^i \right)$$

where $1_{(k=i)}$ is the indicator function for $k = i$. Since $\sigma^s_D \left( s_t^i \right) = \sigma_e + \sigma^s \left( s_t^i \right)$, and since by construction

$$\sum_{k=1}^{n} \left( \frac{\alpha_0 + \alpha_1 s_t^i }{\alpha_0 s_t^i + \alpha_3 s_t^i } \right) 1_{(k=i)} \alpha \phi q K_t^i \alpha_k^i + g_k^i \sigma_k^i \left( s_t^i \right) s_t^i = 1$$

we can rewrite

$$\sigma^s_{R,t} = \frac{S_t^0 \alpha \left( 1 - \lambda S_t^0 \right) \sigma_e + \sum_{k=1}^{n} \left( \frac{\alpha_0 + \alpha_1 s_t^i }{\alpha_0 s_t^i + \alpha_3 s_t^i } \right) 1_{(k=i)} \alpha \phi q K_t^i \alpha_k^i + g_k^i \sigma_k^i \left( s_t^i \right) }{f_1^t \left( \frac{s_t^i}{s_t^i} s_t^i \right)} + \frac{1}{1 + f_2 (S_t^i s_t^i) \left( \frac{s_t^i}{s_t^i} \right) + \eta_{k,t}^i} \sigma^s_D \left( s_t^i \right) + \sum_{k \neq i} \eta_{k,t}^i \sigma^s_D \left( s_t^i \right)$$

where

$$f_1^t \left( \frac{s_t^i}{s_t^i} s_t^i \right) = \frac{\alpha_0 + \alpha_2 \left( s_t^i \right) \left( \frac{s_t^i}{s_t^i} \right) }{\alpha_1 + \alpha_3 \left( s_t^i \right) \left( \frac{s_t^i}{s_t^i} \right) }$$

and

$$f_2 (S_t^i s_t^i) = \frac{\alpha_2 \left( s_t^i \right) + \alpha_3 \left( s_t^i \right) S_t^0}{\alpha_0 + \alpha_1 S_t^0}$$

and

$$\eta_{k,t}^i = \frac{\left( \phi q K_t^i \alpha_k^i + g_k^i \right) s_t^i}{\left( \alpha_0 s_t^i + \alpha_3 s_t^i \right) \left( \frac{s_t^i}{s_t^i} \right) }$$

Note that also that

$$f_1^t < 0 \text{ if and only if } \frac{\alpha_2 \left( s_t^i \right) }{\alpha_3 \left( s_t^i \right) } < \frac{\alpha_0}{\alpha_1} = \frac{1}{\alpha_p \theta_q}$$

Q.E.D.

**Part (c) of Proposition 2: The Expected Return**

The expected return is obtained immediately from $\sigma^s_{R,t}$ by using the formula

$$E_t \left[ dR_t^{TW} \right] = - \text{Cov}_t \left( dR_t^i, \frac{dm}{m_t} \right)$$

Q.E.D.

**Proof of Proposition 1:** (a) The price consumption ratio of the total wealth portfolio can be obtained by simply adding the prices of individual securities. In particular, we find

$$\alpha_0^{TW} \left( s_t^i \right) = \sum_{i=1}^{n} \alpha_0^i s_t^i + \sum_{i=1}^{n} \phi q K_t^i \sum_{j=1}^{n} \alpha_j^k s_t^j = \left( 1 + \phi q \right) \alpha_0^i s_t^i$$

$$\alpha_1^{TW} \left( s_t^i \right) = \sum_{i=1}^{n} \alpha_0^i s_t^i + \sum_{i=1}^{n} \phi q \sum_{k=1}^{n} \left( \alpha_p^k \phi q K_t^i + 5 \alpha_p^k \phi q K_t^i K_t^i + \alpha_p^k \phi q K_t^i K_t^i \right) s_t^i$$
Algebra shows

\[
\alpha^\text{TW}_\text{T} (s_t) = \frac{1}{1 - \phi H_q} \alpha^\prime q s_t
\]

\[
\alpha^\text{TW}_1 (s_t) = \frac{1}{1 - \phi H_q} \left( (\alpha_p s_t) K_p \phi s \alpha_{pq} + \alpha_{pq} s_t \right)
\]

Part (b). An application of Ito’s Lemma to \(P^\text{TW}_t = C_t \left( \alpha^0 \text{TW}_t (s_t) + \alpha^1 \text{TW}_t (s_t) S_t^\gamma \right) \) implies that the diffusion part of the TW portfolios is given by

\[
\sigma^\text{TW}_{P; t} = \sigma_c + \frac{\alpha^1 \text{TW}_t (s_t)}{\alpha^1 \text{TW}_t (s_t) + S_t^\gamma \alpha (1 - \lambda S_t^\gamma) \sigma_c}
\]

\[
+ \frac{\alpha^0 \text{TW}_t (s_t) + S_t^\gamma \alpha^1 \text{TW}_t (s_t)}{\alpha^0 \text{TW}_t (s_t) + S_t^\gamma \alpha^1 \text{TW}_t (s_t)} \sum_{j=1}^{n} \alpha^j \text{TW}_t (s_t) \left( K_p \phi s \alpha_{pq} \alpha_p^j + \alpha_{pq} \right) s_t^j (\nu^j - s^j \cdot \nu)
\]

\[
= \frac{S_t^\gamma \alpha (1 - \lambda S_t^\gamma) \sigma_c}{f^\text{TW}_t (s_t) + S_t^\gamma} + \sum_{j=1}^{n} w^\text{TW}_{jt} \sigma_D (s_t)
\]

with

\[
f^\text{TW}_t (s_t) = \frac{\alpha^0 \text{TW}_t (s_t)}{\alpha^1 \text{TW}_t (s_t)}
\]

and where

\[
w^\text{TW}_{jt} = \frac{\alpha^j \text{TW}_t (s_t) + S_t^\gamma \left( K_p \phi s \alpha_{pq} \alpha_p^j + \alpha_{pq} \right)}{\sum_{k=1}^{n} \alpha^k \text{TW}_t (s_t) + S_t^\gamma \left( K_p \phi s \alpha_{pq} \alpha_p^k + \alpha_{pq} \right)} s_t^j
\]

are weights such that \( \sum_{j} w^\text{TW}_{jt} = 1 \). Given the form of the stochastic discount factor, we obtain

\[
E_t \left[ dR^\text{TW}_t \right] = -\text{Cov}_t \left( dR^\text{TW}_t, \frac{dm}{m_t} \right)
\]

\[
= (\gamma + \alpha (1 - \lambda S_t^\gamma) \left( \frac{f^\text{TW}_t (s_t) + S_t^\gamma}{f^\text{TW}_t (s_t) + S_t^\gamma} \sigma_c + \sum_{j=1}^{n} w^\text{TW}_{jt} \sigma^j \sigma_{C,F,t} \right)
\]

Q.E.D.

**Proof of Proposition 3:** As mentioned, part (a) is shown in MSV. In this case, we have \( a_0 = (p + k + \phi)^{-1} \), \( a_1 = a_0 k \tilde{G} / (p + \phi) \), \( a_2 = a_0 \phi / (p + k) \) and \( a_3 = a_0 \phi k \tilde{G} / p (1 / (p + k) + 1 / (p + \phi)) \), which can also be obtained as a special case of the previous formulas.

(b) The total wealth portfolio P/C ratio in this case does not depend on the shares \( s_t \), but only on the surplus consumption ratio \( S_t \). It follows that the return of the total wealth portfolio is instantaneously perfectly correlated with consumption growth, and thus with the stochastic discount factor. In this case, the conditional CAPM representation holds, and the beta is given by

\[
\beta^\text{DISC} \left( S_t, \frac{\bar{S^\gamma}}{S_t^\gamma} \right) = \frac{\text{Cov}_t (dR_t^\text{DISC}, dR^\text{TW}_t)}{\text{Var}_t (dR^\text{TW}_t)} + \frac{1 + \frac{\gamma \phi \sigma_s}{\phi k \tilde{G} + \phi \sigma_s} \sigma_s (S_t)}{1 + \frac{\phi \sigma_s}{\phi k \tilde{G} + \phi \sigma_s} \sigma_s (S_t)}
\]

(40)

where \( \sigma_s (S) = \alpha (1 - \lambda S) \) and

\[
f^\text{DISC} \left( \bar{S^\gamma} / S_t^\gamma \right) = \frac{\phi (p + \phi) \left( \bar{S^\gamma} / S_t^\gamma \right) + (p + k) (p + \phi)}{\phi (p + \phi) \left( \bar{S^\gamma} / S_t^\gamma \right) + (p + k) (\phi \left( \bar{S^\gamma} / S_t^\gamma \right) + p)}
\]

48
Proof of Proposition 4: Note that $f_{DISC}(1) = 1$, which implies $\beta_{DISC}(S_t, 1) = 1$. In addition, it is easy to see that $f_{DISC}'(\bar{s}_t/s_t) < 0$, which implies that $\partial \beta_{DISC}/\partial (\bar{s}_t/s_t) > 0$. Q.E.D.

Proof of Proposition 5: Part (a) is in Santos and Veronesi (2005). As for part (b), note that the diffusion term of returns is

$$\sigma^i_t = \sigma_c + \frac{s_t^i}{s_t^i + \bar{s}_t^i} \left( \nu^i - \sum_{j=1}^n s_j^i \nu^j \right)$$

which can be rewritten as

$$E_t \left[ dR_t^i \right] = \frac{\sigma_{CP,t}^i + \sigma_c^2 \frac{\sigma_t^i}{\bar{s}_t^i}}{1 + \frac{\sigma_t^i}{\bar{s}_t^i}}$$

Since in this case, $P_t^{TW} = \rho^{-1} C_t$, we immediately find

$$E_t \left[ dR_t^i \right] = \beta^i \left( \bar{s}_t^i / s_t^i \right) E_t \left[ dR_t^{TW} \right]$$

with

$$\beta^i = \frac{\sigma_{CP,t}^i / \sigma_c^2 + \frac{\sigma_t^i}{\bar{s}_t^i}}{1 + \frac{\sigma_t^i}{\bar{s}_t^i}}$$

Q.E.D.
Table I

Basic moments

Panel A: Summary statistics for the market portfolio

<table>
<thead>
<tr>
<th></th>
<th>( \bar{R}_M )</th>
<th>( \text{vol}(R_M) )</th>
<th>( \tau_f )</th>
<th>( \text{vol}(\tau_f) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7.71%</td>
<td>16.25%</td>
<td>1.44%</td>
<td>3.08%</td>
</tr>
</tbody>
</table>

Panel B: Predictability regressions

Panel B-1: Sample 1948-2001

<table>
<thead>
<tr>
<th>Horizon</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ln \left( \frac{D}{P} \right) )</td>
<td>.13</td>
<td>.2</td>
<td>.26</td>
<td>.35</td>
</tr>
<tr>
<td>( t-\text{stat} )</td>
<td>(2.13)</td>
<td>(1.65)</td>
<td>(1.34)</td>
<td>(1.29)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>.09</td>
<td>.10</td>
<td>.11</td>
<td>.14</td>
</tr>
</tbody>
</table>

Panel B-2: Sample 1948-1995

<table>
<thead>
<tr>
<th>Horizon</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ln \left( \frac{D}{P} \right) )</td>
<td>.28</td>
<td>.48</td>
<td>.63</td>
<td>.78</td>
</tr>
<tr>
<td>( t-\text{stat} )</td>
<td>(4.04)</td>
<td>(4.00)</td>
<td>(4.49)</td>
<td>(5.41)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>.19</td>
<td>.32</td>
<td>.43</td>
<td>.54</td>
</tr>
</tbody>
</table>

Panel C: The value premium

<table>
<thead>
<tr>
<th>Portf.</th>
<th>Growth</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \bar{R} )</td>
<td>6.86%</td>
<td>7.77%</td>
</tr>
<tr>
<td>( ME/BE )</td>
<td>5.05</td>
<td>2.68</td>
</tr>
<tr>
<td>( P/D )</td>
<td>43.47</td>
<td>31.38</td>
</tr>
<tr>
<td>CAPM ( \beta )</td>
<td>1.13</td>
<td>1.02</td>
</tr>
</tbody>
</table>

Notes to Table I. Panel A: Summary statistics for the market portfolio. \( \bar{R}_M \) is the annualized average excess returns of the market portfolio over the three month Treasury Bill. \( \text{vol}(R_M) \) is the annualized standard deviation of the returns on the market portfolio. \( \tau_f \) is the average risk free rate, as measured by three-month Treasury Bill rate, and \( \text{vol}(\tau_f) \) is its annualized standard deviation. Panel B: Predictability quarterly regressions of excess returns at the 1, 2, 3, and 4 year horizon on the log of the price dividend ratio of the market portfolio. \( t-\text{stat} \) denotes the Newey-West \( t-\) statistic where the number of lags is the double of the forecasting horizon. Panel C: The value premium. \( \bar{R} \) is the annualized average excess returns of each of the decile portfolios, \( ME/BE \) is the average market-to-book and \( P/D \) the average price dividend ratio. CAPM \( \beta \) is obtained by running time series regressions of excess return on each of the ten decile portfolios sorted on \( ME/BE \) on the market excess return, where \( ME \) is the market equity and \( BE \) is the book value. Returns, dividends, returns, market equity and other financial series are obtained from the CRSP-COMPUSTAT database. The sample period is 1948-2001. The construction of the BE/ME sorted portfolios follows the standard procedure of Fama and French (1992): Each year \( t \) portfolios are sorted into 10 BE/ME sorted portfolios using book-to-market ratios for year \( t-1 \). Returns on each of these portfolios are calculated from July of year \( t \) to June of year \( t+1 \).
Table II
Model parameters used in the simulation

Panel A: Consumption and preference parameters

<table>
<thead>
<tr>
<th>µc</th>
<th>σc</th>
<th>γ</th>
<th>ρ</th>
<th>γ/S</th>
<th>min{γ/S_t}</th>
<th>α</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>.02</td>
<td>.015</td>
<td>1.5</td>
<td>.072</td>
<td>48</td>
<td>27.75</td>
<td>77</td>
<td>.13</td>
</tr>
</tbody>
</table>

Panel B: Share process parameter

<table>
<thead>
<tr>
<th>n</th>
<th>θ_{CF}</th>
<th>\bar{\sigma}</th>
<th>\phi</th>
<th>ν_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>[-.0032, .0032]</td>
<td>.005</td>
<td>.052</td>
<td>.47</td>
</tr>
</tbody>
</table>

Notes to Table II. Panel A: µc is the annual average growth rate of the consumption process, σc is the standard deviation of consumption growth, γ is the coefficient controlling the local curvature of the utility function, ρ is the subjective discount rate, G, λ, α and k are the parameters controlling the dynamics of the process G_t = S_t^{-γ}, where S_t = (C_t - X_t)C_t^{-1} is the surplus consumption ratio and the process for G_t is given by

\[ dG_t = \left[ k \left( \bar{G} - G_t \right) - \alpha(G_t - \lambda) \mu_t \right] dt - \alpha(G_t - \lambda) \sigma_t dB_t. \]

Panel B: The share process for i = 1, 2, ..., n is

\[ ds_t^i = \phi \left( \bar{\sigma}^i - s_t^i \right) + s_t^i \sigma_t dB_t, \]

n = 200 is the number of assets in our artificial economy. θ_{CF} is the parameter controlling the cash-flow risk. Each asset is assigned a value of θ_{CF}, which are distributed uniformly in the range above. \bar{\sigma}^i is the fraction that each asset contributes to consumption in the steady state and \phi is the speed of mean reversion of the share process. Finally, σ_t^i = ν^i - s_t^i ν where ν^i are vectors with ν^1_i = θ_{CF}/σ_c, ν^i_{i+1} = ν^i, and the remaining entries equal to zero. The simulation consists of 10,000 years of daily data.
### Table III
Basic moments in simulated data

Panel A: Summary statistics for the aggregate portfolio

<table>
<thead>
<tr>
<th></th>
<th>$\bar{R}^M$</th>
<th>vol($R^M$)</th>
<th>$\bar{\tau}^f$</th>
<th>vol($\tau^f$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.40%</td>
<td>13.60%</td>
<td>.71%</td>
<td>4.46%</td>
</tr>
</tbody>
</table>

Panel B: Predictability regressions

<table>
<thead>
<tr>
<th>Horizon</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>ln ($\frac{\bar{D}}{\bar{P}}$)</td>
<td>.22</td>
<td>.34</td>
<td>.43</td>
<td>.49</td>
</tr>
<tr>
<td>t−stat.</td>
<td>(30.72)</td>
<td>(37.78)</td>
<td>(44.89)</td>
<td>(52.30)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>.06</td>
<td>.09</td>
<td>.11</td>
<td>.11</td>
</tr>
</tbody>
</table>

Panel C: The value premium

<table>
<thead>
<tr>
<th>Portf.</th>
<th>Growth</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$\bar{R}$</td>
<td>3.03%</td>
<td>3.79%</td>
</tr>
<tr>
<td>$ln (\frac{\bar{P}}{\bar{D}})$</td>
<td>6.62</td>
<td>5.12</td>
</tr>
<tr>
<td>CAPM $\beta$</td>
<td>.88</td>
<td>.92</td>
</tr>
<tr>
<td>CAPM fitted returns</td>
<td>3.90</td>
<td>4.02</td>
</tr>
<tr>
<td>$\text{Avge}(\theta_{CF}^i) \times 100$</td>
<td>−.2705</td>
<td>−.1501</td>
</tr>
</tbody>
</table>

**Notes to Table III.** Panel A: Summary statistics for the market portfolio. $\bar{R}^M$ is the annualized average excess returns of the market portfolio over the three month Treasury Bill. vol($R^M$) is the annualized standard deviation of the returns on the market portfolio. $\bar{\tau}^f$ is the average risk free rate and vol($\tau^f$) is its annualized standard deviation. Panel B: Predictability quarterly regressions of excess returns at the 1, 2, 3, and 4 year horizon on the log of the price dividend ratio of the market portfolio. t−stat denotes the Newey-West t−statistic where the number of lags is the double of the forecasting horizon. Panel C: Annualized average returns $\bar{R}$, average log price-dividend ratio, $ln (\frac{\bar{P}}{\bar{D}})$, and CAPM $\beta$. CAPM fitted returns are the returns resulting from multiplying the CAPM betas from the previous line by the average excess return of the market portfolio reported in Panel A. $\text{Avge}(\theta_{CF}^i) \times 100$ refers to the average $\theta_{CF}^i$ (multiplied by 100) for the assets in the corresponding decile portfolio.
### Table IV

**Asset pricing models: Time series regressions (quarterly)**

**Panel A:** Time series regression \( R_t^i = \alpha + \beta^M R_t^M + \epsilon_t \)

<table>
<thead>
<tr>
<th>Portf.</th>
<th>Growth</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>-0.46</td>
<td>-0.03</td>
</tr>
<tr>
<td>( t(\alpha) )</td>
<td>(-2.00)</td>
<td>(-1.18)</td>
</tr>
<tr>
<td>( \beta^M )</td>
<td>1.13</td>
<td>1.02</td>
</tr>
<tr>
<td>( t(\beta^M) )</td>
<td>(39.80)</td>
<td>(43.68)</td>
</tr>
</tbody>
</table>

**Panel A-2: Empirical data**

<table>
<thead>
<tr>
<th>Portf.</th>
<th>Growth</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>-0.22</td>
<td>-0.06</td>
</tr>
<tr>
<td>( t(\alpha) )</td>
<td>(-16.10)</td>
<td>(-3.03)</td>
</tr>
<tr>
<td>( \beta^M )</td>
<td>.88</td>
<td>.92</td>
</tr>
</tbody>
</table>

**Panel B:** Time series regression \( R_t^i = \alpha + \beta^M R_t^M + \beta^{HML} R_t^{HML} + \epsilon_t \)

<table>
<thead>
<tr>
<th>Portf.</th>
<th>Growth</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>.20</td>
<td>.17</td>
</tr>
<tr>
<td>( t(\alpha) )</td>
<td>(1.13)</td>
<td>(1.05)</td>
</tr>
<tr>
<td>( \beta^M )</td>
<td>1.04</td>
<td>.99</td>
</tr>
<tr>
<td>( t(\beta^M) )</td>
<td>(43.68)</td>
<td>(51.25)</td>
</tr>
<tr>
<td>( \beta^{HML} )</td>
<td>-0.42</td>
<td>-0.12</td>
</tr>
<tr>
<td>( t(\beta^{HML}) )</td>
<td>(-12.13)</td>
<td>(-2.37)</td>
</tr>
</tbody>
</table>

**Panel B-1: Empirical data**

<table>
<thead>
<tr>
<th>Portf.</th>
<th>Growth</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>-0.01</td>
<td>.07</td>
</tr>
<tr>
<td>( t(\alpha) )</td>
<td>(-.79)</td>
<td>(3.94)</td>
</tr>
<tr>
<td>( \beta^M )</td>
<td>.95</td>
<td>.95</td>
</tr>
<tr>
<td>( \beta^{HML} )</td>
<td>-0.28</td>
<td>-0.17</td>
</tr>
</tbody>
</table>

**Panel B-2: Simulated data**

**Notes to Table IV.** Panel A: Time series regressions in empirical (Panel A-1) and simulated (Panel A-2) data of returns on each of the book-to-market sorted portfolios on the market excess return. \( \alpha \) denotes the intercept of the time series regression and \( \beta^M \) the regression coefficient. \( t(\alpha) \) and \( t(\beta^M) \) denote the heteroskedasticity corrected \( t \)-statistic. Panel B: Time series regressions in empirical (Panel B-1) and simulated (Panel B-2) data of returns on each of the book-to-market sorted portfolios on the market excess return and the returns on HML, where \( \beta^{HML} \) is the regression coefficient on HML. The \( t \)-statistics in simulated data have been omitted as they are all well above 100 for the case of the regression coefficients, \( \beta^M \) and \( \beta^{HML} \).
Table V  
Asset pricing models: Fama-MacBeth regressions (quarterly)

Panel A: Empirical data

<table>
<thead>
<tr>
<th></th>
<th>Const.</th>
<th>Mkt.</th>
<th>SMB</th>
<th>HML</th>
<th>Mkt×log(D/P)</th>
<th>Mkt×cay</th>
<th>Adj. $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.69</td>
<td>-2.52</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>11%</td>
</tr>
<tr>
<td></td>
<td>(3.21)</td>
<td>(-1.65)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>.36</td>
<td>1.63</td>
<td>-.31</td>
<td>1.05</td>
<td></td>
<td></td>
<td>80%</td>
</tr>
<tr>
<td></td>
<td>(.23)</td>
<td>(.99)</td>
<td>(-.31)</td>
<td>(2.16)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.72</td>
<td>-.87</td>
<td></td>
<td>1.71</td>
<td></td>
<td></td>
<td>83%</td>
</tr>
<tr>
<td></td>
<td>(2.24)</td>
<td>(-.65)</td>
<td></td>
<td>(2.46)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3.06</td>
<td>-1.37</td>
<td></td>
<td></td>
<td>.06</td>
<td></td>
<td>81%</td>
</tr>
<tr>
<td></td>
<td>(2.48)</td>
<td>(-1.01)</td>
<td></td>
<td></td>
<td>(2.34)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Panel B: Simulated data

<table>
<thead>
<tr>
<th></th>
<th>Const.</th>
<th>Mkt.</th>
<th>HML</th>
<th>Mkt×log(D/P)</th>
<th>Adj. $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-2.44</td>
<td>3.60</td>
<td></td>
<td></td>
<td>89.0%</td>
</tr>
<tr>
<td></td>
<td>(-21.21)</td>
<td>(29.43)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-.18</td>
<td>1.35</td>
<td>1.04</td>
<td></td>
<td>99.5%</td>
</tr>
<tr>
<td></td>
<td>(-.99)</td>
<td>(7.19)</td>
<td>(24.57)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.75</td>
<td>-.76</td>
<td></td>
<td>1.54</td>
<td>99.6%</td>
</tr>
<tr>
<td></td>
<td>(5.08)</td>
<td>(-2.09)</td>
<td></td>
<td>(11.05)</td>
<td></td>
</tr>
</tbody>
</table>

Notes to Table V. Panel A: Fama-MacBeth regressions in empirical data. Line 1, CAPM regressions where Mkt. represents the average excess return of the market portfolio. Line 2, Fama and French (1993) model, where SMB is the return on “small minus big” and HML is the return on “high minus low”. Line 3, conditional CAPM regression where the dividend yield, log(D/P), of the market portfolio is used as a conditioning variable. Line 4 conditional CAPM regression where the variable cay of Lettau and Ludvigson (2001) is used as a conditioning variable. Panel B: Fama-MacBeth regressions in simulated data. $t$-statistic in parenthesis and Adj. $R^2$ is the adjusted $R^2$. 
Table VI
The dynamics of the value premium

Panel A: Annualized average excess returns (%) in empirical data

<table>
<thead>
<tr>
<th>Market-to-book of market portfolio</th>
<th>( \bar{\tau} )</th>
<th>1</th>
<th>10</th>
<th>10-1</th>
<th>( \bar{R}_M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; ( \bar{\tau} )</td>
<td>15%</td>
<td>13.18</td>
<td>23.57</td>
<td>10.38</td>
<td>15.40</td>
</tr>
<tr>
<td></td>
<td>20%</td>
<td>10.57</td>
<td>21.70</td>
<td>11.14</td>
<td>13.41</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>5.51</td>
<td>19.16</td>
<td>13.64</td>
<td>9.89</td>
</tr>
<tr>
<td></td>
<td>30%</td>
<td>6.97</td>
<td>19.49</td>
<td>12.51</td>
<td>10.50</td>
</tr>
<tr>
<td></td>
<td>35%</td>
<td>8.19</td>
<td>18.65</td>
<td>10.45</td>
<td>11.14</td>
</tr>
<tr>
<td>&gt; ( \bar{\tau} )</td>
<td>15%</td>
<td>5.73</td>
<td>10.35</td>
<td>4.62</td>
<td>6.34</td>
</tr>
<tr>
<td></td>
<td>20%</td>
<td>5.95</td>
<td>10.06</td>
<td>4.11</td>
<td>6.31</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>7.31</td>
<td>10.11</td>
<td>2.80</td>
<td>6.99</td>
</tr>
<tr>
<td></td>
<td>30%</td>
<td>6.82</td>
<td>9.32</td>
<td>2.50</td>
<td>6.62</td>
</tr>
<tr>
<td></td>
<td>35%</td>
<td>6.15</td>
<td>8.98</td>
<td>2.83</td>
<td>5.87</td>
</tr>
</tbody>
</table>

Panel B: Annualized average excess returns (%) in simulated data

<table>
<thead>
<tr>
<th>Price-dividend of market portfolio</th>
<th>( \bar{\tau} )</th>
<th>1</th>
<th>10</th>
<th>10-1</th>
<th>( \bar{R}_M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; ( \bar{\tau} )</td>
<td>15%</td>
<td>7.52</td>
<td>17.62</td>
<td>10.10</td>
<td>10.24</td>
</tr>
<tr>
<td></td>
<td>20%</td>
<td>6.42</td>
<td>15.27</td>
<td>8.86</td>
<td>8.79</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>5.71</td>
<td>13.96</td>
<td>8.25</td>
<td>7.90</td>
</tr>
<tr>
<td></td>
<td>30%</td>
<td>5.29</td>
<td>13.29</td>
<td>7.99</td>
<td>7.37</td>
</tr>
<tr>
<td></td>
<td>35%</td>
<td>5.02</td>
<td>12.64</td>
<td>7.62</td>
<td>6.98</td>
</tr>
<tr>
<td>&gt; ( \bar{\tau} )</td>
<td>15%</td>
<td>2.24</td>
<td>7.17</td>
<td>4.93</td>
<td>3.35</td>
</tr>
<tr>
<td></td>
<td>20%</td>
<td>2.18</td>
<td>7.10</td>
<td>4.92</td>
<td>3.28</td>
</tr>
<tr>
<td></td>
<td>25%</td>
<td>2.14</td>
<td>7.00</td>
<td>4.86</td>
<td>3.21</td>
</tr>
<tr>
<td></td>
<td>30%</td>
<td>2.06</td>
<td>6.79</td>
<td>4.73</td>
<td>3.10</td>
</tr>
<tr>
<td></td>
<td>35%</td>
<td>1.96</td>
<td>6.64</td>
<td>4.68</td>
<td>2.98</td>
</tr>
</tbody>
</table>

Notes to Table VI. Panel A: Annualized average excess returns in empirical data of the growth (portfolio 1) and value (portfolio 10) portfolios depending on whether the market-to-book of the market portfolio is below or above the \( \bar{\tau} \) percentile of its empirical distribution. Panel B: Annualized average excess returns in simulated data of the growth (portfolio 1) and value (portfolio 10) portfolios depending on whether the simulated price-dividend ratio of the market portfolio is below or above the \( \bar{\tau} \) percentile of its distribution in simulated data. \( \bar{R}_M \) is the average excess return on the market portfolio in empirical data (Panel A) and simulated data (Panel B).
Table VII
Time series regressions on double sorted portfolios

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\frac{P}{D}$</th>
<th>$\beta_{HF}^HML$</th>
<th>$\frac{P}{D}$</th>
<th>$\alpha$</th>
<th>$\beta^M$</th>
<th>$\beta^{HML}$</th>
<th>$t(\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>L m</td>
<td>8.22</td>
<td>25.68</td>
<td>.19</td>
<td>1.24</td>
<td>.51</td>
<td>6.64</td>
</tr>
<tr>
<td></td>
<td>L h</td>
<td>8.73</td>
<td>21.64</td>
<td>-.01</td>
<td>1.02</td>
<td>1.10</td>
<td>-.30</td>
</tr>
<tr>
<td></td>
<td>M l</td>
<td>3.95</td>
<td>57.14</td>
<td>.11</td>
<td>.89</td>
<td>-.10</td>
<td>4.71</td>
</tr>
<tr>
<td></td>
<td>M m</td>
<td>6.81</td>
<td>56.48</td>
<td>.14</td>
<td>1.19</td>
<td>.27</td>
<td>8.97</td>
</tr>
<tr>
<td></td>
<td>M h</td>
<td>7.89</td>
<td>53.75</td>
<td>.12</td>
<td>1.33</td>
<td>.41</td>
<td>6.69</td>
</tr>
<tr>
<td>Growth</td>
<td>H m</td>
<td>4.45</td>
<td>302.98</td>
<td>.07</td>
<td>.94</td>
<td>.01</td>
<td>7.87</td>
</tr>
<tr>
<td></td>
<td>H h</td>
<td>6.43</td>
<td>154.72</td>
<td>.11</td>
<td>1.16</td>
<td>.24</td>
<td>8.56</td>
</tr>
</tbody>
</table>

Notes to Table VII. We perform a double sort on price-dividend ratios, $P/D$, and (preformation) loading on HML, $\beta_{HF}^{HML}$, in simulated data. Thus, portfolio Lh denotes the portfolio of assets with low price dividend ratios and high preformation loading on HML. The preformation loading is calculated using 5 years of quarterly simulated data prior to the sorting date. $\overline{P/D}$ is the average price-dividend ratio of the assets in the portfolio, $\alpha$, $\beta^M$, $\beta^{HML}$ are the intercept and loadings on the market and HML resulting from the time series regression

$$R_t^i = \alpha + \beta^M R_t^M + \beta^{HML} R_t^{HML} + \epsilon_t$$

$t(\alpha)$ is the intercept’s, heteroskedasticity corrected, $t$–statistic. The $t$– statistics on the loadings on $R^M$ and $R^{HML}$ are omitted as they are well above 100.

Table VIII
Consumption growth predictability in simulated data

<table>
<thead>
<tr>
<th>Horizon (quarters)</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>log(D/P)</td>
<td>-.0001</td>
<td>-.0002</td>
<td>-.0002</td>
<td>-.0002</td>
</tr>
<tr>
<td>t-stats</td>
<td>(-15.32)</td>
<td>(-17.35)</td>
<td>(-14.26)</td>
<td>(-10.86)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>.4%</td>
<td>.6%</td>
<td>.5%</td>
<td>.3%</td>
</tr>
</tbody>
</table>

Notes to Table VIII. Time series predictability regression, in simulated data, of consumption growth on the lagged dividend yield, log(D/P),

$$\Delta c_{t+k} = \alpha + \gamma \log(D/P) + \epsilon_{t+k} \quad \text{for} \quad k = 4, 8, 12, \quad \text{and} \quad 16.$$ 

$t$–statistics are Newey-West $t$–statistics where the number of lags is double the the predictability horizon period.
The top panel plots the steady-state cash flow component of individual assets’ expected return against the unconditional cash flow risk parameter $\sigma_{CF} = E \left[ \text{cov} \left( dD^i / D^i, dC / C \right) \right]$. For each asset, relative share is assumed equal to one, $\bar{s}^i / s^i_t = 1$, and surplus consumption ratio is assumed equal to its steady state value $S_t = \bar{S}$. The bottom panel reports the same quantities, but under a random selection for relative shares $\bar{s}^i / s^i_t$. 
This figure plots the theoretical beta $\beta_i \left( S_t, \bar{S}_i / s_i \right) = \text{Cov} \left( dR^i_t, dR^{TW}_t \right) / \text{Var} \left( dR^{TW}_t \right)$ in the special case where $\gamma = 1$ and no cross-sectional differences in cash flow risk, that is, $\sigma_{CF}^i = \sigma_{CF}^j = \sigma_c^2$. 

The top left panel plots the average price-dividend ratio (y-axis) of P/D sorted portfolios versus their unconditional average return (x-axis) in artificial data, under the assumption that assets have no cross-sectional differences in cash flow risk, $\sigma_{CF}^i = \sigma_{CF}^j = \sigma_c^2$. Under the same assumptions, the bottom left panel plots the “fitted” average return according to the CAPM, i.e. $E[\text{Return}_i^t] = \beta_{CAPM}^i E[\text{Return}^{mkt}]$, on the y-axis against the average return on the x-axis. The two right-hand panels plots the same quantities as the left hand panels, but under the assumption that individual assets have cross-sectional differences in cash flow risk.
The top panel plots the theoretical discount component of individual stock returns plotted against the relative share $\bar{s}/s_i$, which proxies for expected dividend growth. This quantity is computed for various levels of the asset unconditional cash flow risk $\sigma^i_{CF} = E \{ \text{cov}(dD^i/D^i, dC/C) \}$. The middle panel plots the cash flow risk component of stock returns, plotted against the relative share $\bar{s}/s_i$, again for various levels of unconditional cash flow risk. The bottom panel reports the total conditional expected return for individual assets.