CHAPTER 7

AGGREGATION OF MONETARY ASSETS

7.1. Introduction

Monetary policy is related to the behavior of indices of the quantity, "price", and velocity of money. Yet, for such aggregates to be useful, they must have meaning and must be measurable. This raises troublesome methodological questions. What is money? Is it a "good" whose quantity can be measured, or is it just a vector of different characteristics (liquidity, means of payment, etc.)? Do currency and time deposits possess identical "moneyness" so that they can be summed linearly and with equal weights to acquire a meaningful quantity aggregate? If money is a meaningful good, then what is its price? Can more than one monetary aggregate jointly have meaning? In sections 7.2-7.11, we shall explore these issues using the general theory of economic aggregates. Economic indices are also called functional, true, or exact indices. The other well-known class of indices, used in section 7.12, consists of statistical indices, which are intended to approximate functional indices.

Although not previously applied to money demand, the literature on aggregation theory exists precisely for the purpose of providing rigorous and unique answers to the above questions in terms of a single internally consistent approach. That approach builds upon the aggregation implications of the multistage decision theory introduced in sections 1.3, 5.6, and 6.2. In this chapter we discuss the potential usefulness of the theory of aggregates to the joint construction of theoretically meaningful quantity and price indices at multiple levels of aggregation. We shall apply aggregation theory to the aggregation of passbook accounts across institution types and then to nested aggregation over transaction balances. The approach is the analog to that used in Chapter 6, but in terms of direct demand. This chapter, based upon Barnett (1980a), realizes objectives defined in Friedman and Schwartz (1970, pp. 151-154).

We shall show that passbook accounts at different types of institutions are close substitutes and hence can be aggregated linearly. But aggregation
by simple summation is rejected, since the coefficients of the linear function are unequal. Substitutability between passbook savings and transaction balances is found to be low. Hence, nonlinear aggregation is required to approximate the economic index over savings and transaction balances. Our results also suggest that passbook accounts at commercial banks possess greater “moneyness” than passbook accounts in savings and loans or mutual savings banks, since the economic index weights passbook accounts at commercial banks more heavily than passbook accounts at savings and loans or at mutual savings banks. Similarly, transaction balances are far more heavily weighted than our economic passbook account aggregate. We provide similar empirical results relative to time deposits and large certificates of deposit.

Since the beginning of this century, a highly respected and increasingly sophisticated literature has been under development on statistical index number theory. While aggregation theory results in exact aggregator functions depending upon unknown (but estimable) parameters, statistical index number theory results in parameter-free approximations to aggregator functions. Index number theory provides the basis for the index numbers published by almost every governmental agency in the world (other than the central banks). In the latter sections of this chapter we explore the implication of statistical index number theory for the construction of monetary quantity index numbers.

During the past decade there has been much concern about the apparent destabilization of velocity. In fact, the problem arose primarily because of the long-run substitution effect resulting from rising own rates on unregulated monetary assets relative to the own rate on rate-regulated monetary assets. But the value of an economic aggregate (by its definition) cannot change as a result of internal substitution effects. Hence, the money market substitution effects’ destabilizing velocity should be completely internalized by aggregation over the money market.

When any reputable index number formula is used, we find that the velocity of money is increasingly stabilized as the level of aggregation is increased, but the velocity of the usual simple sum index is destabilized by aggregation beyond an intermediate level. Furthermore, we use information theory to compare the information content of Divisia versus simple sum monetary aggregates. We find that the Divisia index dominates the simple sum index, regardless of the monetary components of the index and regardless of the choice of final targets. The gains in information from the simple sum to Divisia index (over the same components) frequently is very large. The simple sum index is severe& defective.
7.2. Objectives

7.2.1. Theoretical and actual practices

Functional (economic) indices can be constructed at multiple levels of monetary aggregation in such a manner that the multiple indices are fully and uniquely nested. As a result, internal contradictions cannot arise at varying levels of aggregation. The functional index number generation process is inherently linked with a money market modeling procedure, so that a full system model of liquid asset demand would result as a byproduct of the nested index number generation procedure.

Suppose we should wish to construct a monetary quantity index as some function of currency, demand deposits, and consumer-type savings and time deposits at commercial banks. We call the resulting index $M_2$. In economic aggregation theory, consumers then must be able to treat $M_2$ as the quantity of a meaningful single good in their decisions. By the definition of a consumer good, consumers must be able to select their desired aggregate quantity of $M_2$ without regard to its composition. The allocation of $M_2$ over its component elements could be accomplished in a later second stage decision, conditionally upon the prechosen aggregate level of $M_2$. Varying the relative quantities of currency and time deposits within $M_2$ while holding the aggregate $M_2$ level constant must not affect consumers’ preferences over any other goods. If this condition is satisfied, consumers can possess stable preferences over $M_2$ and other goods. If $M_2$ is not a good in this fundamental sense, then consumer preferences over $M_2$ and other goods will appear to shift whenever the relative proportions of the components of $M_2$ change.

It can be shown that when a meaningful functional quantity index exists for a consumer, that index itself must possess the known properties of a utility function, and that utility function must possess certain additional special properties (homotheticity and weakly separable nesting within the consumer’s full utility function). Then, when the aggregate quantity index is held constant, the “utility of money” is necessarily held constant independently of its composition. As has been observed by Samuelson and Swamy (1974, p. 568): “The fundamental point about an economic quantity index, which is too little stressed by writers, Leontief and Afriat being exceptions, is that it must itself be a cardinal indicator of ordinal utility.”

The functional quantity index cannot be known exactly without knowledge of the representative consumer’s utility function, since the functional
quantity index depends upon and is defined in terms of consumer preferences. Conventional accounting practices generate meaningful indices only to the degree that those indices imply plausible preference orderings over components. Yet, all current monetary quantity indices are constructed from simple addition of components. This means that if those indices have any economic (as opposed to accounting) meaning at all, the indices have been generated by a utility function for financial assets possessing the same simple unweighted summation form used in constructing the index. But such a utility function requires the goods over which it is defined to be perfect substitutes in identical ratios. In other words, the components of the quantity index must be indistinguishable to the consumer. In the existing simple sum $M_2$, for example, the consumption characteristics of one dollar of currency must be identical to those of one dollar of long-term time deposits. The violation of aggregation theory increases as the level of aggregation increases, since the higher the level of aggregation the less substitutable the components of the aggregates become.

Velocity can have no more meaning than the quantity aggregates relative to which velocity is defined. If we have no theory treating $M_1$, $M_2$, and $M_3$, for example, as goods related behaviorally, then what do we conclude when $M_1$ goes down, $M_2$ goes up, and $M_3$ remains unchanged? We have ambiguously conflicting information.

We frequently seek information about the “price” of money. In various studies the price of money has been viewed as an interest rate, an index of interest rates, the rate of change of prices, the price level, or an index of some subset of those subindices. But we shall prove that the literature on economic quantity indices and on user costs (equivalent rental prices) can be utilized to derive a unique dual theory of implied monetary rental price indices. Once a monetary aggregate has been operationalized, the consumer’s decision modeled, and the consumer’s preference structure estimated, the “price” and the quantity of the aggregate are simultaneously implied. These indices satisfy the accounting identity of equality between expenditure and the product of quantity and price, and the consumer can be shown to behave in a rational manner relative to the good whose price and quantity have been defined. This rationality obtains both relative to the aggregates and relative to their components, and consumers’ decisions at all levels of aggregation are consistent with a single joint rational choice criterion.

Fisher (1922) provided a list of desirable properties for economic price and quantity indices. Frisch (1930) proved that when the number of goods exceeds unity, no index number formula can satisfy all of those properties. However, Samuelson and Swamy (1974, p. 566) have shown that if price
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and per capita quantity data is assumed to fall on neoclassical demand functions (rather than to be unrestricted independent variables, as assumed by Frisch), then the economic quantity and dual price indices "do meet the spirit of all of Fisher’s criteria in the only case in which a single index number of the price of cost of living makes economic sense—namely the ("homothetic") case of unitary income elasticities in which at all levels of living the calculated price change is the same". We use such economic quantity indices.

We see that the theory of quantity and dual price indices can provide unique and meaningful quantity and price indices. But we seek indices corresponding to more than one level of aggregation. Can this be accomplished in an internally consistent manner? In fact, this can be done using a theory of functional structure which has been developed along with and attached to the recent theory of quantity and dual price indices. Under certain assumptions (weakly separable nesting) on preferences, a hierarchy of aggregates is dictated by that theory. Hence, everything we sought above becomes available under appropriate assumptions on preferences. No contradictions arise, and all becomes understandable jointly within a single recursively nested model of consumer portfolio allocation.

In Chapter 6 we provided a means of applying the theory of aggregation to modeling, recursively aggregating, and estimating a food demand sector. The demand model derived and applied in that chapter is the most flexible of the globally integrable demand models currently available. However, the dual price indices implied by that model are not known, and a closed form solution does not exist for the demand functions. In modeling the money market, we should prefer a simpler specification permitting use of the entire theory of functional structure. We adopt such a simpler specification and use the recursive approach presented in Chapter 6.

While exact aggregator functions form the basis for economic aggregation theory, they contain unknown parameters that must be estimated. We use aggregator functions for hypothesis testing and other research purposes and to reveal the implications of aggregation theory in money markets. However, for data construction purposes, parameter-free "statistical" index numbers are preferable. Hence, we also present results with the use of statistical index numbers, and those results lead us to advocate the use of Divisia monetary quantity indices.

7.2.2. Structural change and the utility approach to money demand

The theory of functional structure (see, for example, Blackorby, Primont and Russell, 1978) possesses a known link with the otherwise unrelated
theory of consumption characteristics. Lancaster and Becker postulated that consumer preferences are more revealingly understood in terms of preferences for certain properties (consumption characteristics) of goods (such as nutrition, flavor, etc.) rather than in terms of the market goods themselves (such as candy, yogurt, etc.). Advocates of that approach argue that the production of consumption characteristics (as through internal household activities) from market goods and the consumption of the resulting characteristics produced are distinguishable phenomena which should not be confounded by modeling market goods demand directly, without regard to characteristics production.

This approach is particularly relevant when one suspects that structural changes are occurring in the mode of transformation of goods into characteristics, since structural (or quality) change need not imply changes in preferences for the resulting characteristics. Hence, the demand for characteristics may be stable, although changes in the transformation between goods and consumption characteristics may result in the appearance of unstable goods demand if the characteristics production and consumption processes are not separated. A rigorous and systematic approach to modeling such decision shifts is provided in Chapter 8.

This theory is relevant to our understanding of money markets. The properties of various money market instruments have been changing. Denominations of Treasury bills have varied; changing regulations have varied the properties of savings deposits and resulted in the introduction of certificates of deposit; NOW accounts (negotiable orders of withdrawal) are changing the consumption characteristics of demand deposits – and then there are electronic funds transfer, repurchase agreements, and money market funds. Yet, consumers’ tastes for liquidity, means of payment, store of value, and other such monetary characteristics may not have changed. Only their mode and efficiency of production may have changed. When we seek stability of money demand we frequently think in terms of the demand for these underlying monetary characteristics; we must therefore remove complicating structural shifts in the transformation of monetary instruments into characteristics.

It has been shown that certain assumptions on the Lancaster-Becker theory of consumption characteristics are sufficient to imply the function structure necessary for our recursive money market modeling and index number theory. Hence, under those assumptions all of the theories we have

‘In the United States, NOW accounts are defined to be interest-bearing demand deposits (checking accounts).
discussed above become unified. Shifts in our index numbers have specific meaning in terms of consumption characteristics, and structural change becomes an index number problem rather than an apparent shift in consumer behavior. It can be shown that under those assumptions increasing amounts of structural change in money markets are internalized into index numbers as the level of aggregation is increased. Hence, if the recursive theoretical aggregation approach is carried to relatively high levels of aggregation within the money market, money demand becomes dependent solely upon the demand for consumption characteristics, which are likely to be stable. At each level of aggregation, shifts or trends in the parameters of economic quantity aggregator functions relate to conditional (upon lower level indices) structural change at the corresponding level of aggregation within the money market.

It perhaps is worth noting that the utility approach to money demand modeling (on which the above results depend), is currently the basis for rapidly expanding empirical research in the literature. Consider, for example, Chetty (1969), Bisignano (1974), Diewert (1974b), Par-kin, Cooper, Henderson and Danes (1975), Clements (1976), Donovan (1978), Philips (1978), Offenbacher (1979), and Clements and Nguyen (1979). Early advocates of the utility approach include Friedman and Patinkin. The utility approach is based upon implicit modeling rather than the explicit modeling used in the transactions demand or portfolio analysis approaches. In an economic (rather than empirical) sense, the utility approach is a reduced form approach which models, restricts, and characterizes the results of the consumer's decisions without the need to consider the explicit structure of the decision. While the "true" utility function does not contain money, a derived utility function containing money generally can be acquired, and it is with this derived utility function that we begin. Regarding its existence in the general case, see Arrow and Hahn (1971, p. 350) and Quirk and Saposnik (1968, p. 97).

In conventional utility modeling of consumer goods demand, the structure of household transformation of goods into ultimate consumption characteristics is absorbed into (and lost within) the utility function. Alternative approaches must be used when we seek explicitly to model structural change within that internal household transformation. Similarly, in the utility modeling of money demand we absorb the transactions technology and other aspects of the structure of the consumer's decision within his utility function. The resulting approach has the merit of unifying the modeling of the demand for all money market instruments within a single framework without the need to explore the detailed and different
structures of the consumer’s decisions within each sector of the money market. If the structure shifts, then the model’s parameters will shift.

We need to incorporate explicit structural economic modeling techniques into our approach only when we require explanation or modeling of large parameter shifts. Considering the surprisingly high precision of our estimators, we shall be content in this chapter with the assumption that our model’s underlying structure is stable, and we do not test explicitly for parameter variation.

In this chapter we postulate the existence of a representative consumer, although we argued against that practice in Chapter 3. We use a community utility function because of its usefulness as an approximation, rather than out of any conviction that such a community utility function actually exists. However, there is some empirical and theoretical evidence that under some conditions the behavior of aggregate consumption data may be approximated by a consistent and transitive preference preordering. See Dixon (1975), Maks (1978), and Donovan (1978).

7.2.3. The velocity function

The concept of velocity becomes particularly meaningful when a velocity function can be factored out of the money demand function. Under the assumption of homotheticity of preferences, velocity will be factorable as a nontautological entity within our model. At any level of aggregation the appropriate velocity function will depend upon the dual prices of the corresponding quantity aggregate and of other quantity aggregates within the same branch of the utility tree. Since those price aggregates previously have never been computed for monetary aggregates, the explanatory variables in the factored velocity function are not currently available. Yet, the theory of nested aggregates automatically would generate the factored velocity function along with its dual price arguments at each level of aggregation. The theory has the potential to simplify and unify our understanding of phenomena which otherwise appear to be complex and puzzling.

However, it should be understood that the resulting velocity functions are not equatable with the usual concept of velocity. The “income” concept relevant to our velocity function is the right-hand side of the budget constraint. Depending upon the level of aggregation and our separability assumptions, that “income” could be total expenditure on monetary assets, total expenditure on monetary assets plus consumption goods, or total wealth, but not an index of national income or product.
7.3. The consumer's decision

7.3.1. Intertemporal allocation

In this section we derive the Jorgensonian user cost (equivalent rental price) of monetary assets from a rigorous Fisherine intertemporal consumption expenditure allocation model. Since the model is formulated in discrete time, a structure of assumptions is required regarding the timing of interest rate and price changes and of portfolio transactions. Although we shall not fix the time interval, it could be set at one day, since interest on savings deposits rarely is paid more frequently than daily. For our purposes a daily discrete time model could be viewed as approximating a continuous time model, since our average quarterly data corresponds to a substantially longer period.

We define time period \( t \) to be the time interval \([t, t+1)\), closed on the left and open on the right. Hence, the instant of time \( t \) is included in interval \( t \), but the instant \( t+1 \) is not. We assume that the consumption of goods can proceed continuously throughout any time interval, although our model will use only the total (integral) of that consumption for any time period. Stocks of monetary assets and bonds are constant during each period, and can change only at the end of an interval. Hence, during period \( t \) any changes in holdings occurring at instant \( t+1 \) are not seen until the initial instant of interval \( t+1 \). In short, all portfolio transactions take place at the boundaries between intervals.

Interest on bonds and on monetary assets is paid at the end of each period. Since the end (right-hand boundary) of period \( t \) is included in period \( t+1 \), but not in period \( t \), interest paid for asset holdings during period \( t \) cannot be consumed until period \( t+1 \). Interest rates, prices, and wage rates remain constant within the interior of each period, but can change discretely at the boundaries of periods. Hence, capital gains (or losses) resulting from changes in market bond yields can take place only at the boundaries of periods.

We treat labor supply as exogenously determined, and we assume that labor supplies, \((L_1, \ldots, L_{t+T})\) during all periods of the consumer's planning horizon are blockwise weakly separable from all other arguments of his utility function, so that we can use the subutility function defined only over the other arguments.

Let \( t \) be the current period (or equivalently the instant of time at the start of the period). Let \( T \) be the length of the planning horizon, so that the consumer currently plans through all periods, \( s \), in \( \{s: t < s < t + T\} \).
We now define our variables:

\(x_s\) = vector of per capita (planned) consumption of goods and services (including those of durables) during period \(s\);

\(p_s\) = vector of goods and services expected prices and of durable goods expected rental prices during period \(s\);

\(m_{is}\) = planned per capita real balances of monetary asset \(i\) during period \(s\) \((i = 1, \ldots, n)\);

\(r_{is}\) = the expected nominal holding period (including capital gains and losses) yield on monetary asset \(i\) during period \(s\) \((i = 1, \ldots, n)\);

\(A_s\) = planned per capita real “bond” holdings during period \(s\);

\(R_s\) = the expected one-period holding yield on “bonds” during period \(s\);

\(L_s\) = per capita labor supply during period \(s\); and

\(w_s\) = the wage rate during period \(s\).

As will be seen in the formulation of the consumer’s decision problem below, \(R_s\) is the expected one-period holding (including realized or unrealized capital gains or losses) yield (during period \(s\)) on assets accumulated to transfer wealth between multiperiod planning horizons rather than to yield liquidity or other services during the current period. As a result, \(A_s\) will enter the consumer’s utility function only during period \(s = t + T\), and \(A\) need not necessarily be bond holdings. We use the word “bonds” (also sometimes referred to as the benchmark asset in this context) to simplify exposition. The benchmark asset’s one-period holding yield during period \(s\) is defined to contain all market premiums available for forgoing the services provided by monetary assets. Observe that the holding period used in defining \(R_s\) must equal that of \(r_{is}\), which is a short rate.

We let \(u_t\) be the representative consumer’s current intertemporal, \(T\)-period, utility function. We assume that \(u_t\) is weakly separable in each period’s consumption of goods and monetary assets, so that \(u_t\) can be written in the form

\[
\begin{align*}
    u_t &= u_t(m_t, \ldots, m_{t+T}; x_t, \ldots, x_{t+T}; A_{t+T}) \\
    &= U_t(v_t(m_t), v_{t+1}(m_{t+1}), \ldots, v_{t+T}(m_{t+T}); \\
    & \quad V(x_t), V_{t+1}(x_{t+1}), \ldots, V_{t+T}(x_{t+T}; A_{t+T})
\end{align*}
\]

(7.1)

for some monotonically increasing, linearly homogeneous, strictly quasi-concave functions, \(v_t, v_{t+1}, \ldots, v_{t+T}, V, V_{t+1}, \ldots, V_{t+T}\). The function \(v\) is monotonically increasing and strictly quasiconcave, but not necessarily linearly homogeneous. The function \(U_t\) also is monotonically increasing.
Dual to the functions $V$ and $V'(s=t+1, \ldots, t+T)$, there exist current and planned true cost of living indices, $p_s^* = p_s(p_t)$ and $p_s^* = p_s^*(p_t)$ ($s=t+1, \ldots, t+T$).\(^2\) Those indices will be used to deflate all nominal quantities to real quantities, as in the definitions of $m_{it}$ and $A$, above.

Assuming replanning at each $t$, we write the consumer’s decision problem during each period $s (t \leq s \leq t+T)$ within his planning horizon so as to choose $(m_{it}, \ldots, m_{i+T}; x_{it}, \ldots, x_{i+T}; A_{i+T}) \geq 0$ to

maximize $u_t(m_t, \ldots, m_{i+T}; x_t, \ldots, x_{i+T}; A_{i+T})$

subject to

\[ p_t^* x_t = w_t L_s + \sum_{i=1}^{n} \left[ (1 + r_{t,s-1})p_{t-1}^* m_{i,s-1} - p_t^* m_{is} \right] \tag{7.2} \]

\[ + \left[ (1 + R_{s-1})p_{s-1}^* A_{s-1} - p_t^* A_{is} \right]. \]

The real value of assets carried over (endowed) from the prior planning period is

\[ \sum_{i=1}^{n} (1 + r_{i,t-1})m_{i,t-1} + (1 + R_{t-1})A_{t-1}, \]

and the real value of the consumer’s provisions for later planning periods is

\[ \sum_{i=1}^{n} (1 + r_{i,t+T})m_{i,t+T} + (1 + R_{t+T})A_{t+T}. \]

Let

\[ \rho_s = \begin{cases} 1, & \text{for } s = t, \\ \frac{1}{1 + R_u}, & t+1 \leq s \leq t+T. \end{cases} \]

Then $\rho_s$ is the discount factor for discounting period $s$ transactions. Observe that $\rho_s \neq \prod_{u=1}^{s-1} (1 + R_u)$, since $R_s$ is not paid during $[s, s+1)$, but rather at the start of $[s+1, s+2)$. In problem (7.2), $(m_t, x_t)$ is actual consumption of goods and monetary assets during period $t$, while $(m_{i+1}, \ldots, m_{i+T}; x_{i+1}, \ldots, x_{i+T})$ is planned consumption of goods and monetary assets.

Since we assume replanning at each period and permit $u_t$ to vary over time, the consumer’s behavior is bound only by his decisions regarding current period consumption. Actual consumption patterns need not evolve

\(^2\) For a discussion of the relevant duality theory, see section 7.5. The true cost of living index for a weakly separable block of goods equals expenditure on those goods divided by the (category) indirect utility function for those goods.
in agreement with prior plans. However, further restrictions (stationary preferences, intertemporal strong separability, and constant rate of time preference) could be imposed upon \( u_t \) to ensure that the sequence of current consumption quantities evolves over time in agreement with plans whenever correct expectations exist for all variables that are not under the consumer's control. Agreement between actual and planned consumption paths is not necessary to the estimation of our model.

Solve (7.2) for \( A_t \), and write the resulting equation for each \( s \) between \( t \) and \( t+T \). Then back substitute for \( A_t \), starting from \( A_{t+T} \) and working down to \( A_0 \), always substituting the lower subscripted equation into the next higher one. Completion of the sequence of back-substitutions results in the single wealth constraint:

\[
\sum_{s=t}^{t+T} \left( \frac{p_s}{\rho_s} \right) x_s + \sum_{s=t}^{t+T} \sum_{i=1}^{n} \left[ \frac{p_i^*}{\rho_s} - \frac{p_i^*(1+r_{is})}{\rho_{s+1}} \right] m_{is} + \sum_{i=1}^{n} \frac{p_i^*(1+r_{is+T})}{\rho_{t+T+1}} m_{i+T} + \frac{p_i^*}{\rho_{t+T}} A_{t+T} = \sum_{s=t}^{t+T} \left( \frac{w_s}{\rho_s} \right) L_s + \sum_{i=1}^{n} \left( 1 + r_{i+1} \right) p_{i-1}^* m_{i+T} + (1 + R_{t-1}) A_{t-1} p_{t-1}^*.
\]

(7.3)

The consumer can now be viewed as maximizing utility subject to the single wealth constraint, (7.3).

The left-hand side of the constraint is the discounted value of goods consumption plus the discounted user-cost evaluated monetary asset holdings plus the discounted cost of passing on \( m_{t+T} = (m_{t+T}, \ldots, m_{t+T})' \) to the next planning period plus the discounted cost of passing on \( A_{t+T} \) to the next planning period. The right-hand side is discounted total labor income plus the value of monetary assets passed to this planning period from the last one plus the value of bonds passed on to the start of this planning period from the end of the last planning period.

7.3.2. The user-cost of monetary assets

From (7.3) we see immediately that the user cost (equivalent rental price) of \( m_{is} \) is

\[
\pi_i^* = \frac{p_i^*}{\rho_s} - \frac{p_i^*(1+r_{is})}{\rho_{s+1}}.
\]

(7.4)
Finally, the current period user cost, $\pi_{it}$, of $m_{it}$ reduces to

$$\pi_{it} = \frac{p^*_i (R_t - r_{it})}{1 + R_t}.$$  

It can be shown that $\pi_{it}$ is the monetary asset analog of the well-known Jorgensonian user cost (rental price) of durable consumer goods (see Donovan, 1978). Correcting the formula for taxation, we obtain

$$\pi_{it} = \frac{p^*_i (R_t - r_{it})(1 - \tau_i)}{1 + R_t (1 - \tau_i)}$$  \hspace{1cm} (7.5)$$

where $\tau_i$ is the marginal income tax rate. Observe that financial asset $i$ is a free good if $r_{it} = R_t$, and observe that the current period user costs of financial assets are independent of expectations. We shall use formula (7.5) to compute the user costs of financial assets.

User costs commonly are viewed as the prices of the services of durables rather than of their stocks (see Donovan, 1978). In that interpretation services are assumed to be proportional to stocks, and units of quantities and prices are assumed to have been chosen such that the proportionality constants are one. Hence user-cost evaluated stocks (stocks multiplied by corresponding user costs) are expenditures on the services of the stocks.

It is interesting to observe that although (7.5) does not depend directly upon inflation rates, the nominal interest rates within the formula can be expected to respond to expected inflation rates. Furthermore, since the well-known user-cost formula for nonmonetary durables services does depend inversely upon the expected inflation rate, it follows that the user cost of monetary assets relative to durables increases as the expected inflation rate increases. Hence, consumers will respond to increased inflationary expectations by substituting consumer durables for monetary assets.

7.3.3. Supernumerary quantities

We have not assumed linear homogeneity of $v$ since that assumption would be unnecessarily strong for our purposes and would imply unitary income elasticities. However, in this subsection we assume a form of marginal homogeneity that will be required for aggregation.

We assume that $v$ depends upon $m_{t-1}$ as well as upon $m_t$. This assumption introduces no complications into the earlier sections, since the consumer selected $m_{t-1}$ during the prior planning horizon and hence $m_{t-1}$
is given and fixed during the current horizon. We further assume that there exist constants, \( \delta = (\delta_1, \ldots, \delta_{\gamma}) \)', and a linearly homogeneous function, \( u \), such that \( v(m_i; m_{t-1}) = u(y_i) \), where \( y_i = (y_{ir}, \ldots, y_{i\gamma})' \) and \( y_{it} = m_{it} - \delta_i m_{i,t-1} \). In short, we assume the existence of proportional habit formation in current (but not future planned) consumption. In the language of the habit formation literature, \( y_i \) is supernumerary consumption of monetary assets and \( \delta_i m_{i,t-1} \) is the quantity of monetary asset \( i \) consumed out of habit (independently of current interest rates or income) during period \( t \).

The theoretical implications of habit formation have been considered by Pollak (1976).

From (7.1), (7.3), (7.4), and (7.5), we see that the consumer’s intertemporal decision problem can be rewritten as to choose

\[
(y_t, m_{t+1}, \ldots, m_{t+T}; x_t, \ldots, x_{t+T}; A_{t+T}) \geq 0 \text{ to maximize }
\]

\[
U_t(u(y_t), v_{t+1}(m_{t+1}), \ldots, v_{t+T}(m_{t+T}));
\]

\[
V(x_t), V_{t+1}(x_{t+1}), \ldots, V_{t+T}(x_{t+T}; A_{t+T})
\]

subject to the single wealth constraint

\[
\sum_{s=t}^{t+T} \left( p'_s / p_s \right) x_s + \sum_{i=1}^{n} \pi_{it} y_{it} + \sum_{s=t+1}^{t+T} \sum_{i=1}^{n} \pi_{is}^t m_{is} + \sum_{i=t}^{n} p_{i,t+T} (1 + r_{i,t+T}) m_{i,t+T} A_{i+T} + \rho_{t+T} A_{t+T} = (1 + R_{t-1}) A_{t-1} p_{t-1}^*.
\]

We have now established the model and assumption structure needed to apply aggregation theory to monetary aggregation. If we must use aggregates, a case can be made for accepting whatever assumptions are required to render economic aggregates meaningful. If we cannot accept the assumptions, we have no economic aggregates at all. As Samuelson and Swamy (1974, p. 592) conclude, “one must not expect to be able to make the naive measurements that untutored common sense always longs for;
we must accept the sad facts of life, and be grateful for the more complicated procedures economic theory devises”.

### 7.4. Conditional current period allocation

Our assumptions on the homogeneous blockwise weakly separable structure of the intertemporal utility function, eq. (7.6), are sufficient for consistent two-stage budgeting. Hence by Green’s (1964) theorem 4 it follows that the consumer can maximize utility (7.6), subject to the wealth constraint (7.7) in two stages. In the first stage the consumer selects aggregate monetary asset expenditure (supernumerary expenditure for the current period) and aggregate consumer goods expenditure for each period within his planning horizon and his terminal bond (or other benchmark asset) holdings, $A_t + \Gamma$. The chosen bond holdings are to be carried forward to the start of his next planning horizon. In the second stage he allocates current aggregate monetary asset expenditure and current aggregate consumer goods expenditure over individual current period monetary assets and consumer goods.

The second stage allocation decision over individual current period supernumerary monetary assets is to select $y_i$ to

\begin{equation}
\text{maximize } u(y_i) \\
\text{subject to } \pi_i^* y_i = M_i^*,
\end{equation}

where $\pi_i^* = \pi_i / p_i^*$ is the real current period user cost of monetary asset $i$, $\pi_1^*, \pi_2^*, \ldots, \pi_n^*$, and $M_i^*$ is the real value of aggregate supernumerary monetary asset holdings allocated to the current period in the consumer’s first stage decision. Observe that $\pi_i^* = (R_i - r_i)(1 - \tau_i)/(1 + R_i(1 - \tau_i))$ independently of $p_i^*$.

The choice between the real values, $\pi_i^*$ and $M_i^*$, and the corresponding nominal values, $\pi_i$ and $M_i$, is arbitrary, since $p_i^*$ can be canceled out of each side of the budget constraint in the nominal case. This observation is just a restatement of the well-known homogeneity of demand. We further could multiply the budget constraint through by $[1 + R_i(1 - \tau_i)]/(1 - \tau_i)$ in order to use $R_i - r_i$ as prices. The simplified formulation then would correspond with that of Klein (1974) and Offenbacher (1979).

We model the conditional current period monetary asset allocation decision, (7.8), in sections 7.5-7.9, and we explore its implications for aggregation.
7.5. Preference structure over financial assets

7.51. Blocking of the utility function

Suppose that $y_t$ contains only total transaction balances and passbook savings deposits at three institution types, and we seek to aggregate passbook savings deposits over institution types and to nest that aggregate within an aggregate of all of the components of $y_t$. We partition the vector $y_t$ such that $y_t = (y_{1t}, y_{2t})'$, where $y_{1t}$ is per capita real supernumerary transaction balances and $y_{2t}$ is a vector of per capita real supernumerary passbook account deposits. We correspondingly partition $\pi_t$ and $\delta$ such that $\pi_t = (\pi_{1t}, \pi_{2t})'$ and $\delta = (\delta_1, \delta_2)'$.

We assume that the utility function, $u(y_t)$, can be written in the blockwise weakly separable form

$$u(y_t) = \mu(y_{1t}, u_2(y_{2t})), \quad (7.9)$$

with the function $u_2$ being linearly homogeneous. As discussed below, these conditions are both necessary and sufficient for the existence of the economic aggregates we seek. This conclusion, based upon Green’s (1964) theorem 4, assumes that $y_t$ is held exclusively by consumers. For firms, the analogous conditions would be applied to the production functions.

Backsubstituting (7.9) into (7.6), observe the way in which we have nested weakly separable blocks within weakly separable blocks. We have established a fully nested utility tree. As a result, we can acquire a rational multistage budgeting procedure in which the structured utility function itself defines the relevant theoretical quantity index at each stage, and duality theory defines the corresponding functional price index. Other financial assets (repurchase agreements, money market mutual funds, Treasury bills, commercial paper, etc.) could be included in the analysis by increasing the dimension of $y_t$, partitioning it into more than two subsectors, and blocking $u$ into multiple blocks accordingly.

In the next subsection we elaborate on the multistage budgeting properties of decision (7.8) and the implications for quantity and price aggregation.

7.5.2. Multistage budgeting

Our assumptions on the properties of $u$ are sufficient for the two-stage solution of the decision problem (7.8). We define that two-stage decision in
this subsection. It should be observed that the homogeneity assumption on $u$ could be deleted if we required only differential consistency of the two-stage decision (see Theil, 1980). However, we define and use global consistency below, as is done in economic aggregation theory.

Let $\Pi_{zt}^* = \Pi_2(\Pi_{zt}^*)$ be a function of the user costs $\pi_{zt}^*$. The first stage of the two-stage decision is to select $y_{1t}$ and $Y_{2t}$, to solve

$$
\begin{align*}
\text{maximize} & \quad \mu(y_{1t}, Y_{2t}) \\
\text{subject to} & \quad \pi_{zt}^* y_{1t} + \Pi_{zt}^* Y_{2t} = M_{zt}^*. 
\end{align*}
$$

(7.10)

From the solution of problem (7.10), the consumer determines aggregate supernumerary consumption of real passbook account services, $\Pi_{zt}^* Y_{2t}$.

In the second stage, the consumer allocates $\Pi_{zt}^* Y_{2t}$ over consumption of the services of passbook accounts at individual institution types. He does so by solving the decision problem:

$$
\begin{align*}
\text{maximize} & \quad u_2(y_{2t}) \\
\text{subject to} & \quad v^*; y_{2t} = \pi_{zt} Y_{2t}.
\end{align*}
$$

(7.11)

It follows from Green’s (1964) theorem 4 that there exists some function, $\Pi$, such that the solution for $y_{1t}$ to problem (7.8) is the same as the solution for $y_{1t}$ acquired from the two-stage decision, (7.10) and (7.11), for any theoretically admissible values of $M_{zt}^*$ and $\pi_{zt}^*$. It furthermore can be shown that if we use that function $\Pi_2$ in (7.10) then $Y_{2t} = u_2(y_{2t})$ at the solution values for $Y_{2t}$ and $y_{2t}$ to the two-stage decision. We shall say that $Y_{2t} = u_2(y_{2t})$ is the economic (or functional) quantity aggregate (or index) corresponding (or dual) to the economic (or functional) user-cost aggregate (or index), $\Pi_{zt}^* Y_{2t} = \Pi_{zt}^*(\pi_{zt}^*)$. We shall call $u_2$ the quantity aggregator function, and we shall call $\Pi$ the user-cost (or price) aggregator function.

In general, the quantity aggregator function is the corresponding (category) utility function. We show in the next subsection that the corresponding price (user-cost) index is equal to expenditure, $\Pi_{zt}^* Y_{2t}$, divided by the (category) indirect utility function (induced by the direct utility function, $u_2$).

This two-stage decision process is two-stage budgeting, and can be extended to n-stage budgeting simply by nesting weakly separable blocks within weakly separable blocks, etc. in an analogous manner. The result that follows from such nesting is purely mathematical and need not be related to actual multistage decision processes. We need only observe that the consumer acts “as if” he were making his decision in stages if his preferences are nested. The approach is that previously used in sections 1.3, 5.6, and 6.2.
The price index, $\Pi^*_i$, and the quantity index, $Y_{2t}$, are economic price and quantity indices. As can be seen from problem (7.10), those indices have all of the properties of quantities and prices of actual goods (whether or not aggregates). In particular, observe that the consumer acts as if actual aggregate goods existed. Also, observe that quantity indices depend exclusively upon quantities, and that price indices depend exclusively upon prices. Furthermore, the budget constraint of problem (7.11) shows that the product of a dual price index and its corresponding quantity index always equals actual expenditure on the goods within the aggregate.

### 7.5.3. Duality

A quantity aggregator function and its corresponding price aggregator function are duals. The mathematics of function duals is not the subject of this chapter and will not be discussed in detail. Nevertheless, the reader familiar with classical duality relationships will recognize the foundations for the following observation. We begin with the two-stage decision defined in the previous subsection.

Dual to the (quantity aggregator) function $u_p(y_{pt})$ exists the function $\Pi_p(\pi_{pt})$ such that the identity $u_p(y_{pt})\Pi_p(\pi_{pt})=y_{pt}\pi_{pt}$ will hold whenever $y_{pt}$ is the solution to the dual problem

$$\min_{y_{pt}} y_{pt}\pi_{pt} \text{ subject to } u_p(y_{pt})=k_1,$$

where $k_1$ is a positive constant.

This duality relationship demonstrates that knowledge of the function $u$ is sufficient for determination of the function $\Pi$. Hence, we need only estimate the conditional demand system solving (7.11) to estimate $\Pi_p$ and therefore to compute estimates of the passbook real user-cost index, $\Pi_{pt}^* = \Pi_p(\pi_{pt}^*)$. We thereby can acquire $\Pi_{pt}^*$, without estimating the higher level utility function, $\mu$, of eq. (7.9). Hence, we could treat $\Pi_{pt}^*$ as given and recursively estimate the utility tree from the bottom up. In fact it can be shown that $\Pi_p(\pi_{pt}^*)$ is just real expenditure on passbook account services, $y_{pt}^*\pi_{pt}^*$, divided by the indirect utility function corresponding to $u_p(y_{pt})$.

Since preferences are assumed to be homogeneous of degree one, it follows that the resulting function, $\Pi_p$, depends only upon $\pi_{pt}^*$ (and is independent of expenditure on passbook account services).

The function $\Pi_i$ is homogeneous of degree 1. Hence $\Pi_p(\pi_{pt}^*) = \Pi_p(\pi_{pt})/p_i^*$. As a result, we can compute the real value of the user-cost
price aggregate, \( \Pi_{p}(\pi_{p})/p_{p}^{*} \), by using real user costs, \( \pi_{p}^{*} \), as arguments for \( \Pi \). Thus, our earlier observation that our estimates do not depend upon the use of \( p_{p}^{*} \) is verified.

It is interesting to observe that this nesting process immediately can be carried to a higher level to acquire a user-cost index for the economic aggregate over passbook accounts and transaction balances taken jointly. Since economic quantity aggregates always are utility functions, the quantity aggregate immediately is seen to be \( u_{t} = \mu(m_{1t}, u_{p}(y_{pt})) = \mu(m_{1t}, u_{pt}) \).

We define the dual user-cost index by observing that dual to the function (quantity index) \( \mu(m_{1t}, u_{pt}) \) exists a function (price index) \( \Pi(\pi_{1t}, \Pi_{pt}) = \Pi(\pi_{1t}, \Pi_{p}(\pi_{pt})) \) such that the identity

\[
\mu(m_{1t}, u_{pt}) \Pi(\pi_{1t}, \Pi_{pt}) = m_{1t} \pi_{1t} + u_{pt} \Pi_{pt} = m_{1t} \pi_{1t} + y_{pt} \pi_{pt}
\]

will hold whenever \( (m_{1t}, u_{pt}) \) is the solution to the dual problem

\[
\text{minimize } (m_{1t} \pi_{1t} + u_{pt} \Pi_{pt}) \text{ subject to } \mu(m_{1t}, u_{pt}) = k_{2},
\]

where \( k_{2} \) is a positive constant.

By Fisher’s factor reversal test (equality of expenditure to the product of the price and quantity index), the price (user-cost) index dual to a functional quantity index must equal total expenditure on the aggregated assets divided by the indirect category (conditional) utility function defined on those assets. Because of our linear homogeneity assumption on category utility functions, total expenditure cancels out of the quotient leaving a functional price index depending solely upon prices.

### 7.6. Recursive estimation approach

The consumer is viewed as making his budgeting decisions from the top of the tree down, as he decentralizes his budgeting to lower levels of aggregation; but we can estimate the entire implied model recursively from the bottom up. We begin at the bottom of the tree and estimate the most disaggregated demand decisions. We compute the implied price (user-cost) and quantity indices, based upon the utility functions we have estimated, and we then move up to estimate the next level using the just-computed price aggregates as instrumental variables. This approach to the recursive estimation of utility trees has been developed by Barnett (1977a), Fuss (1977), and Anderson (1979). Our data consists of quarterly average values from the first quarter of 1970 to the first quarter of 1978. The data sources are described in section 7.7.
Recall that the current period monetary asset allocation problem, (7.8), is defined conditionally upon the consumer price index, $p^*_t$, which is dual to (and therefore derivable from) the consumer goods current period utility function, $V$. Hence, to apply this instrumental variables approach most fully, we should estimate the function, $V$, defined over the consumer goods sector, prior to estimating $u_t$, defined over the monetary asset sector. But aside from $p^*_t$, we seek no other information from the consumption sector. Hence, the cost of strict adherence to the recursive instrumental variables approach is excessive in the case of computation of $p^*_t$.

As a result, we use a statistical index rather than a functional index for $p^*_t$. Statistical price indices can depend upon quantities as well as prices, but cannot depend upon unknown parameters. We assume that $V(x_t) = (x^T B x_t)^{1/2}$ locally for some square matrix $B$ of unknown parameters. That specification can provide a quadratic approximation to any aggregator function. Diewert (1976b) has shown that if a representative consumer exists, then the Fisher Ideal statistical price index (geometric mean of the Laspeyres and Paasche indices) is always equal to the true value of the functional index, $p^*_t$, regardless of the values of the parameters in the matrix, $B$. We shall use the Fisher Ideal price index for $p^*_t$. In computing the Fisher Ideal index, we use the Bureau of Labor Statistics’ CPI as the Laspeyres index and the Commerce Department’s Implicit Price Deflator as the Paasche Index. Some approximation error exists in the use of the CPI as the Laspeyres index, although the error is small (see Triplett, 1976).

Having computed $p^*_t$, we begin our empirical ascent up the utility tree. Recalling the form of eq. (7.1), we begin by estimating $u_2$. Then $u_2(y_{2t})$ becomes the economic quantity index used with $y_{1t}$ in the next (higher) stage. We compute the implied price index dual to $u_2$ and estimate the demand system generated by $\mu$. The procedure could be carried to any level of aggregation, but will be terminated at $\mu$.

7.7. Data

7.7.1. Data sources

Our data consists of quarterly average values from the first quarter of 1970 to the first quarter of 1978. The data sources follow.

In converting nominal balances to per capita balances, we used Census Bureau population data. For the maximum available yield, $R_t$, we used the
maximum of Moody’s A seasoned corporate bond yield and the commercial paper rate. All yields and interest rates were divided by four to acquire quarterly rates of return. Our measure of $R_t$ is consistent with the common convention. See, for example, Offenbacher (1979) and Klein (1974).

In addition to convention, two bodies of theory exist that are relevant to the measurement of $R_t$. Recall from section 7.3 that $R_t$ is the maximum available expected one-period holding yield. Term structure theory, perfect arbitrage, and rational expectations theory jointly imply that $R_t$ should be the maximum available short rate (plus a probably small “liquidity” premium). However, recent empirical research does not support that conclusion. For example, see Shiller (1979) whose results support our measurement method. Shiller found that when the yield curve is upward sloping, the expected one-period holding yield is at least as high as the long rate. In addition, our experiments with alternative measures of $R_t$ indicated considerable robustness. The reason evidently is that $R_t$ appears in all users costs, and hence relative prices are more sensitive to the own rates, $r_t^c$, than to $R_t$.

Commercial bank consumer passbook account deposits were acquired by subtracting Christmas club accounts, business savings, and domestic government savings, NOW accounts, and savings of banks and of foreign official institutions from savings deposits at all commercial banks based upon reported member bank data and estimated nonmember data. Unpublished internal daily average Board data was used to acquire quarterly averages.

Savings and loan association passbook deposits were acquired using Federal Home Loan Bank Board savings and loan association balance sheet data. Since passbook deposit data is available separately from time deposits only for Federally insured S&Ls, we multiplied the total of passbook deposits and time deposits for all S&Ls by the ratio of passbook deposits to the total of time plus passbook deposits at insured S&Ls to acquire an estimate of passbook deposits at all S&Ls. The data was monthly average data acquired by averaging end-of-month data from succeeding months. Quarterly averages were then constructed.

Passbook account deposit data at mutual savings banks were acquired from the National Association of Mutual Savings Banks’ Balance Sheet of Mutual Banks. The data consists of monthly averages computed as averages of succeeding month-end values. Quarterly averages were constructed.

The commercial bank passbook account interest rate was acquired from the “Survey of Time and Savings Deposits” reported in the Federal Reserve Bulletin. The data reflects a one-day survey taken near the end of the first month of the quarter. The nature of the survey question is such that the data can be taken directly as quarterly averages.
The savings and loan association interest rate was acquired from the Home Loan Bank Board’s *Interest and Dividend Practices* survey. The data reflects a one-day survey taken near the end of the first month of the quarter. The mutual savings banks passbook interest rate was acquired from the FDIC quarterly survey, reflecting end of the first month values.

The tax rate was acquired by dividing the sum of Federal personal income tax liability and state and local government personal income tax and nontax payments by the sum of that numerator plus disposable personal income (accrual basis). Although this provides an average rate rather than the marginal rate required by the theory, all of our results are invariant to the values used for the marginal tax rate. This conclusion follows from the homogeneity property of demand, or by dividing both sides of the budget constraint in (7.8) by \((1 - \tau_i) / (1 + K_i A_i(1 - \tau_i))\) and redefining the resulting variable on the right-hand side.

Transaction balances were computed to equal \(M_1\) plus NOW accounts (at all institution types) plus share drafts at credit unions plus demand deposits at mutual savings banks. Internal weekly average Board data was used to acquire quarterly averages.

### 7.7.2. Data transformations

Prior to estimation of the model, the data was transformed to provide normalized user-cost prices and to rescale the data to be closer to 1.0. In this subsection we present those elementary data transformations.

We took the fourth quarter of 1973 as the base quarter for our price indices. We divided each user cost, computed in accordance with eq. (7.3), by the user cost for passbook account services from the same institution type in the base quarter. The transformed user-cost price series thereby equaled 1.0 for each institution type in the base period. In order to ensure that the product of price and quantity remained unchanged by our rescaling, we correspondingly multiplied each of our per capita real passbook balance series by the original base period user cost for the same institution type. We thereby acquired new per capita “quantity” values, defined to equal expenditure evaluated in index period user-cost prices.

We then rescaled the newly transformed per capita quantity data by dividing all of those new quantity series by a common constant. The common constant was the average value of all of the transformed per capita passbook savings quantity values (averaged over all quarters and all institution types). It should be observed that these data rescalings have no
effect on the economics of the model. The objective was to increase computing precision during estimation by avoiding unnecessarily large or small data values.

7.8. Estimation of passbook branch

7.8.1. CES specification

In the current subsection we present our specification for the conditional demand for passbook savings, which is the solution to decision (7.11). Since \( m_{2i} \) is a vector, we implicitly have segmented passbook deposits into categories. We let \( m_{2i} = (m_{21i}, m_{22i}, m_{23i})' \), where \( m_{21i} \) = real per capita holdings of commercial bank passbook accounts, \( m_{22i} \) = real per capita holdings of savings and loan passbook accounts, and \( m_{23i} \) = real per capita holdings of mutual savings bank passbook accounts. We then write the \( t \) th period supernumerary real per capita holdings in passbook account category \( i \) as \( y_{2it} = m_{2it} - \delta m_{2i,t}^{t-1} \).

To clarify our notation, we replace the subscript 2 with \( p \) (for “passbook”). Then \( u_p(y_{pi}) = u_2(y_{2i}), \) etc. The CES specification for \( u_p \) is

\[
u_p(y_{pi}) = \left[ \sum_{i=1}^{3} \alpha_i y_{pi}^\beta \right]^{1/\beta}
= \left[ \sum_{i=1}^{3} \alpha_i (m_{pi} - \delta m_{pi,t}^{t-1})^\beta \right]^{1/\beta},
\]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3)' \) and \( \beta \) are parameters satisfying \( \beta < 1 \) and \( \alpha \geq 0 \). While more flexible utility functions exist than the CES, they did not appear to be appropriate to our objectives. Our approach estimates a demand system that is integrable to a marginally homothetic utility function and has known closed form representations both for the demand system and for the utility function. The model also should be a generalization of the simple sum utility function which provides the conventional quantity indices. The CES satisfies all of those objectives and is a very substantial generalization of the simple sum function. Since the simple sum aggregate is widely used, it could be impractical (at this stage of research) to consider a quantity index more general than the CES. Furthermore, the use of a common elasticity of substitution appears reasonable with our passbook savings data. At higher levels of aggregation, a more flexible functional form would be required.
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In decision (7.11) we let $E_{p1}^* = \Pi_{2t}^* Y_{2t}$, which is total user-cost-evaluated expenditure allocated to passbook account services, determined from the prior allocation stage (one level higher in the utility tree).

The solution to (7.11) is the demand system

$$m_{pi t} = \delta_{pi t} m_{pi t - 1} + \frac{\bar{\alpha}_i \pi_{pit}^* \bar{\beta}}{\pi_{pit}^* \sum_{k} \bar{\alpha}_k \pi_{pkt}^* \bar{\beta} - \pi_{pkt}^* \delta_{pk t} m_{pk t - 1}}, \quad (7.12)$$

where $\bar{\alpha}_i = \alpha_i^{1/(1-\beta)}$ and $\bar{\beta} = \beta/(\beta-1)$, with $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) > 0$ and $\bar{\beta} < 1$.

The vector of parameters $\bar{\alpha}$ is not jointly identified, since the demand system is homogeneous of degree zero in $\bar{\alpha}$. Hence, we impose the identifying restriction $\sum_i \bar{\alpha}_i = 1$. We do so by estimating (7.12) with the normalization $Z_{ij} = 1$, and then renormalizing the resulting estimates to get $\sum_i \bar{\alpha}_i = 1$. The choice of normalization is arbitrary; we can renormalize at will.

We seek to estimate (7.12) in a form that will impose all theoretical restrictions. We do so by transforming the parameters into other parameters that are free of inequality restrictions. We then impose our restrictions by substitution. We can acquire the maximum likelihood (MLE) estimates of the transformed parameters and then acquire the unique MLEs of the original restricted parameters by using the invariance property of the MLE. In particular, we substitute the transformation $\bar{\alpha}_j = y_j^2 (j=1,2,3)$ to impose $\bar{\alpha}_j > 0$, and we estimate the unrestricted parameters $\gamma = (\gamma_1, \gamma_2, \gamma_3)'$. Since $\bar{\beta} < 1$ defines an open set, that restriction (or any other such strict inequality restriction) cannot be imposed. We replace $\bar{\beta} < 1$ with the approximation $\bar{\beta} < 0.9$. We then substitute the transformation $\bar{\beta} = 1.9 - \cosh \theta$ into (7.12) and estimate the unrestricted parameter $\theta$.

Since $y_{pi t} > 0$, it follows that for any $i$, we must have $m_{pi t} > \delta_{pi t} m_{pi t - 1}$ for all $t$. Since passbook deposits never changed by more than 20 percent between quarters in our data, a sufficient condition for that inequality would be $\delta_{pi t} < 0.8$ for all $i = 1, 2, 3$. We shall impose that sufficient condition. In addition, we require that $\delta_{p} > 0$. Although theory does not require this restriction, the logic of the multistage budgeting process becomes more difficult to interpret when $\delta_{p}$ contains negative elements. In addition, our prior views on $\delta_{p}$ impute low probability to negative elements of $\delta_{p}$, and we have seen in Chapter 6 that negative estimates of $\delta_{p}$ tend to have low precision and hence to be statistically indistinguishable from zero at conventional levels of significance. We jointly impose all of these restrictions on $\delta_{p}$ by substituting the transformations $\delta_{pit} = 0.4 (1 + \sin \phi_i)$ for $i = 1, 2, 3$ and estimating the unrestricted vector $\phi = (\phi_1, \phi_2, \phi_3)'$. 
Multiplying (7.12) by $\pi_{pli}^*/E_{p_1}^*$ to acquire the desired expenditure shares, $w_{pli}^* = \pi_{pli}^* m_{pli}^*/E_{p_1}^*$, and making all of the parameter substitutions described above, we acquire our model for the consumer’s desired expenditure shares. Since adjustment costs may exist, we permit actual expenditure shares, $w_{pli}$, to differ from desired expenditure shares, $w_{pli}^*$, in accordance with the partial adjustment scheme $w_{pli}^* = \lambda w_{pli}^* + (1 - \lambda) w_{pli-1}^*$, where $0 < \lambda < 1$. We use the same adjustment rate, $\Lambda$, for each institution type to ensure that the budget constraint will be satisfied in actual expenditure shares as well as in desired budget shares. In addition, equality of adjustment rates appears plausible for passbook accounts at different institution types. Performing all of these transformations on (7.12) we have our passbook deposits allocation model. We take $w_{pli}, i = 1, 2, 3,$ as endogenous and $E_{p_1}^*$ and $\pi_{pli}^*, i = 1, 2, 3,$ as exogenous. We adopt a conventional additive error structure without serial correlation. Serially correlated disturbances did not appear to be a potential problem, since our specification contains lagged values both of quantity demanded (through habit formation) and of expenditure shares (through partial adjustment).

7.8.2. Theoretical index number properties

We now consider the properties of the functional price and quantity index numbers for passbook savings, when aggregation over institution types is to be consistent with the CES consumer preferences specified in the previous subsection.

The functional quantity index is the utility level itself. Normalizing the index to equal 1.0 at the first observation, we acquire the normalized functional quantity index

$$Q_p(y_{pt}) = u_p(y_{pt})/u_p(y_{p_1}).$$

The nominal functional price index that is dual to our CES specification of $u_p$ is

$$\Pi_p(\pi_{pt}) = \left( \sum_{i=1}^{3} \alpha_i \pi_i \right)^{1/\beta},$$

where $(\alpha, \beta)$ are as defined in the previous section. The corresponding normalized nominal user-cost price index is $P_p(\pi_{pt}) = \Pi_p(\pi_{pt})/\Pi_p(\pi_{p_1})$.

A functional quantity index must be linearly homogeneous in its arguments. While $u_p$ is linearly homogeneous in $y_p$, $u_p$ is not homogeneous in $m_p$ unless $\delta_i = 0$ for all $i$. Hence, $u_p$ cannot strictly be viewed as an aggregator function for $m_p$ when some $\delta_i$ is nonzero, although $u_p(y_{pt})$ is always the functional quantity aggregate for the supernumerary quantities, $y_{pl}$. 
The corresponding real price indices are $\Pi_p(\pi_p^t)$ and $P_p(\pi_p^t)$. If we were to require an index of total (rather than per capita) supernumerary nominal balances, we could compute $Q_p(y_p^t)$ using the total passbook deposit data in place of the per capita real balances, $m_p^t$, in the definition of $y_p^t$. The result would be identical to computing $Q_p(y_p^t)$ with population and $p^*$ fixed at index year levels, since those fixed index year levels would be cancelled out of the numerator and denominator of $Q_p(y_p^t)$.

We seek to consider the limiting case in which $\alpha_1 = \alpha_2 = \alpha_3$ and $\beta = 1$. In that case the functional quantity index equals the simple sum of its components. Since the elasticity of substitution, $\sigma$, equals $1/(1 - \beta)$, we see that $\sigma \to \infty$ as $\beta \to 1$. Hence, the special case we are considering is that of three “goods” (or, more appropriately, services) that are perfect substitutes in equal proportions, i.e. indistinguishable goods. When $\beta = 1$ (but the $\alpha_i$ values are not necessarily equal), the functional quantity index acquires the form of a Laspeyres-type (fixed weight linear) quantity index. The functional price index that is dual to the Laspeyres quantity index is the Leontief price index, $\pi_p(y_p^t) = \min \{ \pi_{p}^{ii}/\alpha_i ; i = 1,2,3 \}$. See Samuelson and Swamy (1974, p. 574). Hence, if the monetary quantity index is the usual simple sum index (so that $Q_\alpha = \alpha_1 = \alpha_2 = \alpha_3$), then the corresponding price index is just the minimum user cost.

7.8.3. Results with passbook savings

The parameter estimates for eqs. (7.12) using passbook data and joint maximum likelihood (FIML) estimation are displayed in table 7.1 with standard errors in parentheses and with $\gamma_3$ normalized to equal one. The estimates of $\phi_1$ and $(\phi_2, \phi_3)$ imply boundary solutions for $\delta_{p1}$ and $(\delta_{p2}, \delta_{p3})$ at their lower and upper bounds, respectively. Transforming back to the original parameters of $u_p(y_p)$, we find that the implied joint maximum likelihood estimates are $\beta = 0.62$ and $\alpha = (0.55,0.26,0.20)^t$, where $\alpha$ has been renormalized such that $\sum_{j=1}^3 \alpha_j = 1$.

Precisions (t-ratios) are generally high. The implied elasticity of substitution, $\sigma$, equals 2.66, which is very high. This elasticity is the short-run

<table>
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<th>$\theta$</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\phi_3$</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
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<td>1.43</td>
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<td>(0.27)</td>
<td>(0.18)</td>
<td>(0.06)</td>
<td>(0.06)</td>
</tr>
</tbody>
</table>

Note: Standard errors are in parentheses.
Aggregation of monetary assets

elasticity of substitution, as is relevant to the aggregator function and hence to aggregation and index number theory. With regard to the long-run utility function, see Pollak (1976). We can see just how high that elasticity is by observing that $\sigma$ is monotonically increasing in $\beta$, and $\beta$ must lie between $-\infty$ and 1. Clearly, $\beta = 0.62$ is very close to the upper bound of 1, at which the utility function (and hence the functional quantity index) is linear and demand functions become set valued correspondences. Observe from $\hat{\lambda}$ that the estimated quarterly adjustment rate from desired to actual shares is about 21 percent.

Thus, we see that passbook accounts at different institution types are highly substitutable, and a simple linear quantity index may be a reasonable approximation to the theoretical quantity index. However, the simple sum index requires equal weights in the linear index, and $\hat{\alpha}_2$ differs substantially from $\hat{\alpha}_3$, which does approximately equal $\hat{\alpha}_3$. The tail area of the asymptotic likelihood ratio test of equal $\alpha$ values is less than 0.00001. Since that tail area is well below 0.05, we reject the hypothesis of equal $\alpha$ values. To test the hypothesis of a simple sum aggregate, we should test the hypothesis that $\beta = 1$ jointly with the hypothesis of equal intensity parameters ($\alpha$'s). However, the likelihood function is not uniquely defined when $\beta = 1$, since demand functions become set valued in that case. Hence, a likelihood ratio test is not applicable.

A functional quantity index measures the quantity of a properly aggregated economic “good”. Since $\alpha_1$ clearly exceeds $\alpha_2$ or $\alpha_3$, we see that commercial bank passbook accounts contribute more heavily to that meaningful economic “good” than mutual savings bank or savings and loan passbook accounts. An explanation may lie in the fact that commercial bank passbook accounts possess all of the basic consumption characteristics of the other two types, but greater liquidity through the “one-stop-banking” property made available during routine trips to the bank to deposit funds into checking accounts. Aggregation theory does not attach a name (such as “moneyness” or “liquidity”) to the functional quantity index. However, our use of user costs dictates that the quantity index is the quantity of services provided by the components of the aggregate. Hence, it may not be unreasonable to deduce that commercial bank passbook accounts appear to provide greater “monetary services”

---

5If $\alpha_1 = \alpha_2 = \alpha_3$ with $\beta = 1$, then $u(x_p)$ is a linear function of the usual simple sum index, $\sum_{t=1}^{3} \alpha_{t} m_{p,t}$. But with unequal $\alpha_i$ values, our economic quantity index is a linear function of $\sum_{t=1}^{3} \alpha_{i} m_{p,t}$, not of the simple unweighted sum.
than passbook accounts at the other two institution types. If funds were transferred from savings and loan passbook accounts to commercial bank passbook accounts, our functional quantity index would increase, perhaps to reflect the economy’s increased liquidity. The usual sum index would not change.

We also observed that computed values of the normalized functional quantity index, $Q_p(y_{pl})$, and the normalized user-cost price index, $P_p(\pi_{pl})$, tended to move in opposite directions, as would be expected from movement along a demand curve. This result is not surprising since Regulation Q cannot decrease the user cost of passbook account deposits to below the equilibrium price, although the regulation can raise the user cost to above the equilibrium level. Hence, an excess supply but not an excess demand can exist in the passbook account market. We therefore can expect the data always to lie on the demand function, even when the market is out of equilibrium. In addition, governmental rate setting tends to minimize simultaneous bias in estimators that condition upon exogenous user costs.

There appears to be information contained in the fact that $\delta_{p1}$ is at its lower bound, while $\delta_{p2}$ and $\delta_{p3}$ are large. Recall that $\delta_{pl}m_{plt-1}$ is a vector of quantities consumed out of habit (or for “subsistence”) regardless of the variations in user costs or in total consumption expenditure within the sample period. Evidently commercial bank passbook accounts contain actively managed primary balances, while mutual savings bank and savings and loan passbook accounts contain a greater percentage of less actively managed secondary balances and saved consumer reserve funds.

When integrability conditions are imposed, as we have done, it is common for some of them to be binding. Hence, the existence of binding regularity conditions is not surprising. Nevertheless, it is also possible that the boundary solutions on the habit formation parameters may have resulted from the joint use of habit formation dynamics and partial adjustment dynamics. Despite the fact that all of the model’s parameters are identified, the data may not contain sufficient information to permit distinguishing adequately between the two sources of dynamic consumer behavior.

7.9. Transactions balances

7.9.1. Specification

We now progress to the next level of the utility tree in (7.8) to estimate $\mu$. We again use a CES utility function. At this level of aggregation it no longer would be reasonable to assume that elasticities of substitution are constant between all monetary assets. But we now have only two “goods”
and hence only one elasticity of substitution. The flexibility of the CES specification therefore still remains satisfactory for our purposes. Furthermore, a constant finite elasticity of substitution, even between all monetary assets, would be more reasonable than the uniformly infinite elasticities of substitution implied by the usual simple sum indices.

We specify $\mu$ to be CES in two goods: real per capital supernumerary transactions balances, $y_{1t}$, and the economic real per capita supernumerary passbook savings aggregate, $u'_{p1} = u_p(y_{p1})$. We introduce no additional habit formation at this level of aggregation (in $y_{1t}$ and the aggregate $u'_{p1}$), since habit formation is already built into $u_p(y_{p1})$ through the specification of $y_{p1}$, and since we expect short-run Engel curves in $m_{1t}$ to pass through the origin. Observe therefore that $y_{1t} = m_{1t}$ and that $\mu$ is homothetic in real per capita transaction balances and in aggregate real per capita supernumerary (not total) passbook savings deposits.

Offenbacher’s (1979) results suggest that currency and demand deposits do not satisfy the conditions for aggregation ‘by summation; however, separate treatment of those two components requires imputation of separate own rates to each. In this chapter we avoid such ambiguous and controversial imputations. Hence, we condition upon summed transaction balances as an elementary good.

We impute to $m_{1t}$, the user-cost price, (7.5), with the own rate set equal to zero. We impute to the supernumerary passbook aggregate, $u_p(y_{p1})$, the dual user-cost functional price index, $\Pi_p = \Pi_p(\pi_{p1})$. We do not introduce adjustment dynamics at this level of aggregation. Since the turnover rates of transaction balances are high, we believe that adjustment to the desired transaction balances share in monetary asset consumption is rapid.

Combining both stages of the decision over transaction balances and passbook savings deposits, we find that consumers are viewed as allocating expenditure over transaction balances and passbook savings deposits (either jointly or through the equivalent two-stage decision) by utility maximization (with habit formation in passbook savings preferences) to acquire desired consumption levels. The desired level of transaction balances is then purchased without lags. In addition, the desired level is acquired of current total user-cost-evaluated expenditure on the services of passbook savings deposits, but its distribution over institution types differs from the desired allocation in accordance with the linear partial adjustment mechanism used in section 7.8.

The utility function is of the CES form

$$\mu(m_{1t}, u_{p1}) = \mu(m_{1t}, u_{p1}) = \left(\alpha_1 m_{1t}^\beta + \alpha_2 u_{p1}^\beta\right)^{1/\beta},$$

where $(\alpha_1, \alpha_2, \beta)$ are parameters satisfying $\beta < 1$ and $(\alpha_1, \alpha_2) > 0$. 
The conditional decision problem at this level of aggregation is to choose \((m_{1t}, u_{pt})\) to

maximize \(\mu(m_{1t}, u_{pt})\)

subject to \(m_{1t} \pi_{1t}^* + u_{pt} \Pi_p(\pi_{pt}^*) = E_t^*\),

(7.13)

where \(E_t\) is user-cost-evaluated expenditure allocated to the services of real transaction balances and of real supernumerary passbook savings deposits during the current period.

We define the expenditure share of transaction balances in \(E_t^*\) to be \(w_{1t} = m_{1t} \pi_{1t}^*/E_t^*\). The share of supernumerary passbook deposits then is \(w_{pt} = 1 - w_{1t}\). After employing parameter transformations analogous to those in section 7.8, we find that the solution to (7.13) can be written in the form

\[
\gamma_1 \pi_{1t}^* = \gamma_1 \pi_{1t}^* (1.9 - \cosh \psi)
\]

and \(w_{pt} = 1 - w_{1t}\), where \(\Pi_{pt}^* = \Pi_p(\pi_{pt}^*)\).

Let \(\Pi_{pt}^*\) be the value of \(\Pi_p(\pi_{pt}^*)\) with the parameters of \(\Pi_p\) replaced by their estimates acquired in section 7.8. We replace \(\Pi_{pt}^*\) with \(\hat{\Pi}_{pt}^*\), normalize \(\gamma_2\) to equal 1.0, and estimate (7.14) with an additive disturbance term. Fuss (1977) has considered the properties of such nested estimation procedures.

Letting \(\epsilon_t (t = 1, \ldots, T)\) be the additive error in equation (7.14) we introduce first-order autocorrelation by specifying that \((\epsilon_2, \ldots, \epsilon_T)\) is a sample from a stationary scalar autoregressive stochastic process satisfying the stochastic difference equation \(\epsilon_t = \rho \epsilon_{t-1} + u_t\), where the sequence \((u_t; t = 2, \ldots, T)\) consists of independently and identically distributed normal random variables with mean zero. The same value, \(\rho\), is used in defining the error structure for each of the two demand equations derived from (7.13). That procedure follows from Berndt and Savin (1975), when no serial correlation of disturbances exists across equations. The parameter \(\rho\) is subject to the constraint \(-1 < \rho < 1\). To impose that restriction, we let \(\rho = \sin \psi\). We eliminate that equality by substitution and estimate the unconstrained parameter, \(\psi\).

To estimate (7.14) with the additive autoregressive disturbance, \(\epsilon\), we use the following transformation. Let the right-hand side of (7.14) be written as \(f(\pi_{1t}^*, \Pi_{pt}^*; \gamma_1, \theta)\), so that

\[
w_{1t} = \rho w_{1t-1} + [f(\pi_{1t}^*, \Pi_{pt}^*; \gamma_1, \theta) - \rho f(\pi_{1t-1}^*, \Pi_{pt-1}^*; \gamma_1, \theta)].
\]

If we add \(\epsilon_t\) to the right-hand side of (7.14), it then follows that the
disturbance to be added to the right-hand side of the transformed equation is $e_t - \rho e_{t-1} = u_t$. So we can estimate the transformed equation using maximum likelihood estimation with a conventional disturbance, $u_t$.

### 7.9.2. Estimates

The resulting maximum likelihood estimates of $(\gamma, \theta, \psi)$ are presented in table 7.2. Transforming back to the original parameters of $u$, we find that $\beta = -2.53$, $\hat{\beta} = 0.96$, and $(\hat{\alpha}_1, \hat{\alpha}_2) = (0.77, 0.23)$, where $(\alpha_1, \alpha_2)$ have been renormalized to sum to one. Our estimate of the intensity parameter, $\alpha_1$, is more than three times our estimate of $\alpha_2$. Hence, we might deduce that transaction balances, $m_{1t}$, contribute to our monetary asset economic quantity aggregate more heavily than our nested passbook deposits aggregate, $u_{pt1}$. However, one should be cautious about viewing the intensity parameters as simple weights in this case, since $\mu$ is a nonlinear function rather than a linear weighted average.

The implied elasticity of substitution is $1/(1 - \hat{\beta}) = 0.28$. Substitutability between transaction balances and passbook savings deposits is far lower than between passbook accounts at different institution types. The elasticity of substitution of 0.28 is too low and the precision of its estimator is too high to justify a linear approximation (requiring infinite elasticity of substitution) to $\mu$.

### 7.9.3. Functional index numbers

In the present section our highest level aggregator function is $\mu$. Hence, our highest level economic quantity aggregate is $u_t = \mu(m_{1t}, u_p(y_{pt}))$. The nominal dual user cost aggregate is

$$\Pi(\sigma_{1t}, \Pi_{pt}) = (\tilde{\alpha}_t^{\beta_{1t}} + \tilde{\alpha}_2^2 \Pi_{pt}^{\beta})^{1/\beta},$$

where $\tilde{\alpha}_t = \alpha_t^{\beta/(1-\beta)}$ and $\beta = \beta/((\beta - 1)$.
In summary, we have acquired the following nested pair of quantity and nominal dual user cost indices, with all indices normalized to equal 1.0 in the first quarter. For passbook accounts we have the maximum likelihood estimate of the normalized functional quantity index, $Q_p(y_{pt})$, and its nominal dual user cost index, $P_p(\pi_{pt})$. For our higher level ($M_2$-type) monetary asset aggregate, we have the maximum likelihood estimate of the normalized functional quantity index, $Q(m_{1t}, y_{pt}) = \mu(m_{1t}, u_p(y_{pt}))/\mu(m_{11}, u_p(y_{p1}))$, and its nominal dual user-cost index, $P(\pi_{1t}; \Pi_{pt}) = \Pi(\pi_{1t}; \pi_p(\Pi_{pt}))/\Pi(\pi_{11}, \Pi_p(\pi_{p1})).$

7.9.4. Implications of estimates

While passbook accounts at different institutions are excellent substitutes, we find no evidence to support equal weighting of the accounts across institutions. Although a simple linear (Laspeyres-type) index of passbook deposits may be useful, the conventional unweighted sum index should be understood to be based upon accounting practice rather than upon any economically meaningful index number construct. If one sought no more than total dollar deposits in passbook accounts in all institution types, the use of simple summation would be dictated tautologically by an accounting identity.

The simple sum index in economics corresponds to the degenerate limiting special case of preferences having linear indifference curves at 45° angles, and the corresponding dual price index is the poorly behaved Leontief fixed coefficients index. In our case, consumers would use passbook accounts in only one institution type, unless all institutions paid the exact same interest rate. If all institutions did pay the exact same interest rate, then the budget constraint would lie on top of a linear indifference curve, and consumers would not care how they allocated funds over institution types. No unique solution would exist. But in fact commercial banks pay lower interest rates than the other two institution types yet acquire stable nonzero deposits. Since passbook accounts across institution types do provide very similar services, we should expect to find even poorer support for the simple sum index at higher levels of aggregation within the money market, and that conclusion generally is supported by our results with transaction balances at the next aggregation level.

When we pass to a higher level of aggregation to incorporate transaction balances into our monetary aggregate, the possibility of a useful linear approximation, even with unequal coefficients, disappears. Transaction
balances and passbook savings are not perfect substitutes and possess an elasticity of substitution of only 0.28. The usual simple sum monetary quantity index is rejected. The current $M_2$ aggregate provides useful accounting information on commercial bank liability structure, but is not well designed as an economic monetary quantity index.

7.10. Empirical selection of blocking

7.10.1. Conditions on elasticities of substitution

In section 7.5 we selected our homogeneous weakly separable blocking of the current period conditional utility function, $u$, on a priori grounds. That blocking then dictated the components of each subindex and index at all levels of aggregation within our hierarchy of aggregates. Conditionally upon that blocking we determined, in sections 7.8 and 7.9, that the form of the aggregator function over the components of each index precludes the use of aggregation by simple summation. In the current subsection we briefly consider the possibility of formally testing for the blocking itself, rather than solely for the form of the preblocked utility (aggregator) function.

We begin with the current period monetary asset utility function, $u(y_t)$, for the vector of real supernumerary per capita holdings, $y_t$, of all monetary assets in the economy. We seek a partitioning, $y_t = (y_{1t}, \ldots, y_{Mt})$', such that $u$ can be written in the blockwise weakly separable form,

$$u(y_t) = u_1(y_{1t}), u_2(y_{2t}), \ldots, u_M(y_{Mt}),$$

(7.15)

with $u_k$ linearly homogeneous for all $k = 1, \ldots, M$. The existence of such a homogeneous weakly separable blocking is necessary and sufficient for the existence of consistent quantity aggregation (to the functional quantity aggregates, $u_1(y_{1t}), u_2(y_{2t}), \ldots, u_M(y_{Mt}))$. Clearly, our earlier a priori blocking, (7.9) was a special case of (7.15) with one-dimensional $y_{1t}$ and with $M = 2$.

Necessary and sufficient conditions for that homogeneous weakly separable blocking are that the elasticity of substitution between any component of $y_{kt}$ (for fixed $k = 1, \ldots, M$) and any (supernumerary) monetary asset not in $y_{kt}$ be independent of the element of $y_{kt}$ selected. We shall refer

6 The conditions could be substantially weakened by dropping the homogeneity condition, if we permit Fisher's factor reversal test to be violated.
to those conditions on elasticities of substitution as the aggregation conditions. Systematic testing for those conditions with monetary assets has not yet been undertaken and is a promising area for future research. However, Appendix E contains elasticity of substitution estimates (without formal separability hypothesis tests) between many categories of monetary assets. The conclusions suggested (at unknown statistical significance levels) by comparisons of those elasticity of substitution estimates follow.

7.10.2. Empirical evidence

The estimates in Appendix E indicate the following. Over the past decade substitutability among passbook accounts at the three institution types (commercial banks, S&Ls, and MSBs) has risen substantially and to high levels. In addition substitutability is high between small time deposits at S&Ls and MSBs. However, substitutability is low between time deposits at commercial banks and at either of the two thrift institutions. Those individuals who purchase small time deposits at commercial banks evidently perceive them to possess properties that are, in some ways, significantly different from those of small time deposits at S&Ls or MSBs. This result is not surprising, since those individuals who purchase small time deposits at commercial banks generally are locked into the lower yields paid by the commercial banks, as a result of the penalty structure imposed on early redemption. In fact it would be difficult to understand why anyone would hold commercial bank small time deposits if he considered them to be close substitutes for small time deposits at thrift institutions. In general, substitutability within the many diverse groups of financial assets considered in Appendix E has tended to rise over the past decade. However, with the exception of the two cases just described, substitutability between financial assets has remained very low.’

We now consider the implications of those elasticity of substitution estimates for the selection of the components of aggregates. In Appendix E we find that the elasticities of substitution between passbook accounts at different institution types are far higher than the elasticities of substitution between passbook accounts at any one of those institution types and any

*Earlier published studies of substitutability between monetary assets have all indicated very low substitutability between monetary assets. Hence, our results are in general agreement with the earlier findings, and our finding of current high substitutability between passbook accounts at the three institution types and between small time deposits at thrift institutions are thereby strengthened by contrast.*
other financial asset. Hence, any aggregate (such as the old $M_2$ index) which contained passbook accounts at some but not at all institution types would violate the aggregation conditions. Similarly, we find that any aggregate containing small time deposits at S&Ls must also contain small time deposits at MSBs. In short, the empirical evidence in Appendix E tends to support aggregation of like-assets over institution types, as proposed in Barnett, Beck, Ettin, Kalchbrenner, Lindsey, Porter, Simpson, and Tinsley (1979).

In sections 7.8 and 7.9 we considered the separate question of whether aggregation over given components can be accomplished by simple summation. Aggregation by summation is a special case of linear aggregation. The necessary and sufficient conditions on elasticities of substitution for linear aggregation are infinite elasticities of substitution between all components within the aggregate. We call those conditions the linearity conditions. The frequently very low elasticities of substitution found in Appendix E further strengthen our rejection of the linearity conditions in sections 7.8 and 7.9.

It should, however, be observed that our inferences drawn from Appendix E, without formal statistical testing, are highly tentative. Our conclusions in this subsection should be viewed as suggestive of areas for future research through systematic hypothesis testing with models specifically designed for that purpose.

7.11. Statistical index numbers

7.11.1. Definition

In previous sections we have been using aggregation theory. In aggregation theory, aggregator functions are utility functions for consumers and production functions for firms. Aggregator functions provide the foundations of aggregation theory, and hence their existence and properties are important in understanding aggregation. By estimating aggregator functions in previous sections, we have acquired information regarding the components of consistent aggregates, and we have determined that aggregator functions defined over financial assets cannot be adequately approximated by simple summation. Aggregation theory itself then would leave us with the alternative of using the actual nonlinear aggregator function in aggregating over monetary assets.
However, as we have seen, functional quantity aggregators depend upon the quantities of the component goods and upon unknown parameters. Estimates of the unknown parameters depend upon the specified model, the data, and the estimator. Hence, aggregator functions, although important in theory and in hypothesis testing, are not generally useful in constructing index numbers which are publishable as data by governmental agencies. For precisely that purpose, the theory of statistical index numbers has been developed. We introduce and then use that highly practical theory in this section.

A functional quantity aggregator depends only upon component quantities and unknown parameters. Functional quantity aggregators cannot depend upon prices, and the definition of a functional quantity aggregator does not depend upon maximizing behavior by economic agents. On the other hand, statistical index numbers do not depend upon any unknown parameters, but quantity index numbers can depend upon component prices as well as upon component quantities, and the definition of exact statistical index numbers does depend upon the maximizing behavior of economic agents. In brief, the introduction of prices (and maximizing behavior in the exact case) into index number theory permits us to dispense with the unknown parameters that exist in the aggregator functions. The merits of the resulting index numbers are not dependent upon any specialized properties of the aggregator function (such as linearity of the function).

A quantity index between periods \( t-1 \) and \( t \), \( Q(\pi_{t-1}, \pi_t; m_{t-1}, m_t) \), is a function of the vectors of prices (user costs) in periods \( t-1 \) and \( t \), \( \pi_{t-1} > 0 \) and \( \pi_t > 0 \), and the corresponding quantity vectors, \( m_{t-1} > 0 \) and \( m_t > 0 \). Diewert defines such an index to be exact for a given aggregator function, \( f \), if \( Q(\pi_{t-1}, \pi_t; m_{t-1}, m_t) = f(m_t)/f(m_{t-1}) \) whenever \( m_t > 0 \) is the value of \( m > 0 \) which maximizes \( f(m) \) subject to \( \pi'_m < \pi_t/m_t \). In other words, a quantity index number is exact if it exactly equals the aggregator function whenever the data is consistent with microeconomic maximizing behavior. Since the aggregator function depends only upon quantities, the index number is a quantity index number despite the existence of prices in its formula. Given a quantity index, the corresponding price index then can be computed from Fisher’s weak factor reversal test. See Diewert (1976b, p. 115).

The form of the index numbers does not depend upon whether the aggregator function is a utility function or a production function. If distributional data were available on shares held by firms (versus households) or by different categories of wealth holders, that information could be incorporated directly into the index number. See Theil (1967, ch. 5) for
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Two particularly noteworthy contributions exist in the recent literature on index numbers. Hulten (1973) has proved that in continuous time the Divisia index is always exact for any consistent (blockwise homothetically weakly separable) aggregator function. Hence, no index number can be better than the Divisia in continuous time. The Divisia index is the line integral defined by the differential $d\log Q = \sum_{i=1}^{N} s_i \ d\log q_i$, where $s_i = \frac{p_i x_i}{p' x}$. Although no always-exact index numbers are known in the discrete time case, Diewert (1976b) has constructed an elegant theory of superlative index numbers in discrete time. Diewert defines an index number to be “superlative” if it is exact for some aggregator function, $f_s$, which can provide a second-order approximation to any linearly homogeneous aggregator function. We call such an index number Diewert-superlative.

Fisher (1922) advocated the following quantity index number, called the Fisher Ideal index:

$$Q_r^F = Q_{r-1}^F \left[ \frac{\left( \sum_{i=1}^{N} \pi_{it} m_{it} \right) \left( \sum_{i=1}^{N} \pi_{i,t-1} m_{i,t-1} \right)}{\left( \sum_{i=1}^{N} \pi_{it} m_{i,t-1} \right) \left( \sum_{i=1}^{N} \pi_{i,t-1} m_{i,t-1} \right)} \right]^{1/2}.$$

Tornquist (1936), and subsequently Theil (1967), advocated the following quantity index number, called the Tornquist-Theil Divisia index:

$$Q^T_T = Q_{T-1}^T \prod_{i=1}^{N} \left( m_{it} / m_{i,t-1} \right)^{(1/2)(s_{it} + s_{it-1})},$$

where

$$s_{it} = \pi_{it} m_{it} / \sum_{k=1}^{N} \pi_{kt} m_{kt}.$$

Taking logarithms of each side, observe that

$$\log Q^T_T - \log Q^T_{T-1} = \sum_{i=1}^{N} s_{it}^* (\log m_{it} - \log m_{i,t-1}),$$

(7.16)

where $s_{it}^* = (1/2)(s_{it} + s_{it-1})$. The same index numbers result, regardless of whether the aggregator functions are utility functions or production functions.

Diewert (1976b) has proved that both the Fisher Ideal and Tornquist-Theil Divisia indices are Diewert-superlative. In addition, as can be seen
from (7.16), the Tornquist-Theil Divisia index provides a discrete time approximation to the optimal continuous time Divisia index. In fact the Tornquist-Theil Divisia index can be derived by numerical integration of the Divisia line integral. The Tornquist-Theil Divisia index and the Fisher Ideal index are highly reputable throughout all segments of the current literature on index numbers, both for their statistical and economic properties. As a quantity index the Tornquist-Theil Divisia index is more widely used than the Fisher Ideal index, since eq. (7.16) permits a natural interpretation of the index. Observe that the growth rate of the index is a weighted average of the growth rates of the components. The weights are the share contributions of each component to the total value of the services of all components. Because of the availability of that transparently clear interpretation, we advocate use of the Tornquist-Theil Divisia index to measure the quantity of money at all levels of aggregation (at least at levels higher than M₃).

7.11.2. Example

In this section we consider the case of an aggregate having the following components: transaction balances, passbook savings at the three institution types and at credit unions, small time deposits at the three institution types, and negotiable and non-negotiable large CDs at commercial banks. The components were selected on the basis of ready availability of the data rather than as a proposal. The proper procedure for selecting components was described in section 7.10, but we seek only an example in the current section. The collection of components will be called $M_3$. Table 7.3 displays the GNP velocity of the Tornquist-Theil Divisia index, of the Fisher Ideal index, and of the simple sum index for seasonally adjusted data. Velocity is normalized to be one in the first quarter. Observe that the velocities of the Fisher Ideal and Tornquist-Theil Divisia indices are identical to three decimal places, so that the choice between those two indices is of no importance.

This phenomenon resulted from the fact that each is a Diewert-superlative index number. Hence, if an aggregator function exists and if maximizing behavior obtains, then the two indices can differ only by a third-order remainder term. In addition, each of the indices should agree

\[\text{For further details on Divisia indices see Barnett (1980c).}\]
Table 7.3

GNP velocities of three monetary quantity index numbers (seasonally adjusted data).

<table>
<thead>
<tr>
<th>Quarter</th>
<th>Fisher</th>
<th>Ideal</th>
<th>Törnquist–Theil</th>
<th>Divisia</th>
<th>Simple sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1968(1)</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>1968(2)</td>
<td>1.014</td>
<td>1.014</td>
<td>1.014</td>
<td>1.014</td>
<td>1.014</td>
</tr>
<tr>
<td>1968(3)</td>
<td>1.013</td>
<td>1.013</td>
<td>1.013</td>
<td>1.013</td>
<td>1.013</td>
</tr>
<tr>
<td>1968(4)</td>
<td>1.008</td>
<td>1.008</td>
<td>1.008</td>
<td>1.008</td>
<td>1.008</td>
</tr>
<tr>
<td>1969(1)</td>
<td>1.017</td>
<td>1.017</td>
<td>1.017</td>
<td>1.017</td>
<td>1.017</td>
</tr>
<tr>
<td>1969(2)</td>
<td>1.031</td>
<td>1.031</td>
<td>1.031</td>
<td>1.031</td>
<td>1.031</td>
</tr>
<tr>
<td>1969(3)</td>
<td>1.052</td>
<td>1.052</td>
<td>1.052</td>
<td>1.052</td>
<td>1.052</td>
</tr>
<tr>
<td>1969(4)</td>
<td>1.057</td>
<td>1.057</td>
<td>1.057</td>
<td>1.057</td>
<td>1.057</td>
</tr>
<tr>
<td>1970(1)</td>
<td>1.062</td>
<td>1.062</td>
<td>1.062</td>
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<td>1.062</td>
</tr>
<tr>
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<td>1.057</td>
<td>1.057</td>
<td>1.057</td>
<td>1.057</td>
<td>1.057</td>
</tr>
<tr>
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<td>1.047</td>
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<tr>
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<td>1.023</td>
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</tr>
<tr>
<td>1971(1)</td>
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with the unknown aggregator function equally as well as they agree with each other, since the remainder term is of the same order in either case.

However, the ordinary simple sum index differs **substantially** from the two Diewert-superlative indices. In addition, the range of values of the velocity of the sum index (0.201) is more than twice that of the superlative indices (0.089). The velocity of the simple sum index (labeled “$M_3$ simple sum”) and of a Diewert-superlative (labeled “$M_3$ Diewert-sup”) index are plotted in fig. 7.1. The Diewert-superlative indices are too close to be plotted separately.

The velocity of the simple sum index continues declining secularly from 1972(3), while the velocity of the Diewert-superlative index rises. Our aggregate does not include many money market instruments such as RPs, treasury bills, commercial paper, money market funds, etc. while our aggregate includes many assets subject to governmental rate regulation. Hence, we should expect substitution’ (disintermediation) to occur out of our aggregate and into such substitutes during periods of rising interest rates and high inflation, if our $M_3$ index approximates an economic monetary good. In such cases velocity should **rise**. Clearly the declining velocity of the simple sum index is very misleading.

![Figure 7.1. Seasonally adjusted velocity (normalized).](image)
Comparing fig. 7.1 with the ten-year government bond rate in fig. 7.2, we see that variations in the velocity of the Diewert-superlative index make economic sense; the interest elasticity of money demand has the correct sign. Internalizing further money market substitution by aggregating over further money market instruments can be expected to further stabilize the velocity of the superlative index. The substitution effect (defined to hold utility constant) of a change in the relative prices of components within an aggregate cannot change the value of an economic quantity aggregate (utility level)!

In contrast, the trend in velocity of the simple sum index would suggest that, in response to rising interest rates and rising inflationary expectations, monetary asset holders have increased the fraction of GNP allocated to consumption of the services of the lowest yielding (largely rate controlled) sector of the market. Disintermediation thereby would appear (misleadingly) to have proceeded within the money market in the wrong direction;

\[ \text{Since GNP does not include the user-cost evaluated services of durables or of monetary assets, our conclusion is based upon the use of GNP as an approximation to the corresponding theoretical national product concept.} \]
It is not surprising that simple sum aggregates frequently provide conflicting information.

It is tempting to conclude that the reason the velocity of the Diewert-superlative index tracks the government bond rate is the fact that the Diewert-superlative index depends upon interest rates. However, the index is constructed to approximate the aggregator function, which depends only upon quantities and therefore not upon $R_t$. The computational reason for the divergence between the Diewert-superlative and sum indices can be seen from eq. (7.16). The Tornquist-Theil Divisia index (or therefore, approximately, any Diewert-superlative index) weights transaction balances more heavily than any of the other components of the aggregates, since transaction balances provide the largest share of monetary services, $s_{it}$. An economic reason for the heavy weighting of transaction balances is that their liquidity contributes heavily to monetary services. But the velocity of transaction balances has been rising rapidly in recent years. Hence, the inadequate weighting of transaction balances in the simple sum $M_3^+$ has permitted velocity to be drawn down by the substitution effect of the increasing relative price (user cost) of transaction balances relative to less liquid monetary substitutes.

To further verify our interpretation, we now incorporate elements of the unregulated money market into $M_3$ to create $M_3^+$. We incorporate dealer and directly placed commercial paper, repurchase agreements (RPs) of commercial banks with the nonbank public, bankers’ acceptances, and negotiable Treasury securities with less than one year remaining to maturity. In fig. 7.1 we plot the velocity of $M_3^+$, with $M_3^+$ computed as a simple sum index (labeled “$M_3^+$ simple sum”), as a Diewert-superlative index (labeled “$M_3^+$ Diewert-sup”), and as a chained Laspeyres index (labeled “$M_3^+$ Laspeyres”). We continue to normalize all velocities to equal 1.0 in the first quarter.

Clearly internalizing those additional segments of the money market has further stabilized the velocity of the Diewert-superlative index. The velocity of the simple sum index continues to trend in the wrong direction. The Laspeyres index is seen to provide a far better approximation than the simple sum index, despite the fact that the Laspeyres index provides only a first-order approximation to the value of the aggregator function. The slight variations remaining in the velocity of the Diewert-superlative index continue to correlate with the ten-year bond rate and to reflect the fact that some elements of the unregulated money market remain outside of the aggregate.

An entirely rigorous conclusion would be based upon the observation that the velocity of the Diewert-superlative index reveals (to the second
order) movements along the aggregator function and therefore movements of the underlying economic aggregate. Hence, fig. 7.1 indicates that the velocity of the simple sum index has been moving in the wrong direction, in the sense of moving in the direction opposite to that of the economic aggregate.

The simple sum index is a Laspeyres quantity index with the weights erroneously set to be equal. Clearly the erroneous weighting destroys the index’s critical independence of substitution effects (within the aggregate), and hence the simple sum index cannot approximate the economic aggregate.

7.12. Information content of the index

In this section we apply information theory to compare (Törnquist–Theil) Divisia monetary quantity indices with the conventional sum indices. In each case we compute the information that the monetary aggregate provides about relevant common policy targets. The section is based upon Barnett and Spindt (1979).

Let the state of the economy in period $t$ be summarized by the $n$-dimensional vector, $s_t$. Its components are defined to contain final policy target variables and the per capita growth rate of a monetary aggregate. At time $t-1$, $s_t$ has not yet been generated by the economy and is a random vector, $S_t^*$, determined from the economy’s reduced form. In this section only, we emphasize that distinction by using capital letters for random variables and corresponding lower case letters for realizations. Let $S_t^*$ be partitioned such that $S_t^* = (X_t, Y_t')'$, where $Y_t$ is the $n-1$-dimensional vector of policy target variables and $X_t$ is the per capita growth rate of a monetary aggregate. Let $f(s_t)$ be the joint density of $S_t^*$, let $g(Y_t)$ be the marginal density of $Y_t$, and let $h(Y_t|X_t)$ be the conditional density of $Y_t$, given $X_t = x_t$.

We explore the information about $Y_t$ that would be acquired by conditioning upon knowledge of $X_t$. The expected information content, 

$$I_{Y_t|X_t} = H_{Y_t} - H_{Y_t|X_t},$$

about $Y_t$ from knowledge of $X_t$, is the reduction in expected uncertainty (entropy). The reduction is from the unconditional values, $H_{Y_t}$, to the conditional value, $H_{Y_t|X_t}$, where $H_{Y_t} = E(-\ln g(Y_t))$ and $H_{Y_t|X_t} = E_{X_t}[E_{Y_t}(-\ln h(Y_t|X_t))|X_t]$. The information function, $I_{Y_t|X_t}$, is zero

10 See Theil (1967). The subscript on the expectations operator identifies the random variables with respect to which the expectation is being taken.
valued if and only if $Y_t$ and $X_t$ are stochastically independent.” We assume that the marginal distribution of $S_t$ is multivariate normal at each $t$.

Let $\sigma_{XX}^2$ be the variance of $X_t$, let $\Omega_{YY}$ be the contemporaneous covariance matrix of $Y_t$, and let $\Omega_{SS}$ be the contemporaneous covariance matrix of $S_t$. Then it follows, under normality assumption, that

$$z_{Y|X} = \frac{1}{2} \log \frac{|\Omega_{YY}| |\sigma_{XX}^2|}{|\Omega_{SS}|}$$

at each $t$. We now estimate $I_{Y|X}$ under various definitions of $Y_t$ and $X_t$, and under two different assumptions on the stochastic process generating $S_t$.

### 7.12.1. Sample estimates

In this subsection we make the strongly simplifying assumption that the moments of $S_t$ do not vary over time, so that the maximum likelihood estimate of $I_{Y|X}$ can be computed directly from the empirical distribution function of the data by using the corresponding sample moments in equation (7.17). We compute the resulting maximum likelihood estimates of the information content, $I_{Y|X}$, of several monetary quantity aggregates with respect to the three definitions of $Y_t$ described in table 7.4. The six sets of components of the monetary aggregates considered are defined in table 7.5. For purposes of comparison, the monetary aggregate in each case is computed both as a conventional simple sum and as the (Törnquist–Theil) Divisia quantity index.

In table 7.6 the percentage gain in information content in going from the simple sum to the Divisia index are reported. Except in one case, the Divisia index dominates the sum index, regardless of the selection of targets, $Y_t$, or of the selection of components for the monetary aggregates.

### 7.12.2. Extensions

Two areas for further research are particularly promising. We could permit the state vector to contain intertemporal components. In addition we could weaken the constant-moments assumption contained in the previous sub-

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11 See Tinsley Spindt and Friar (1980). Also, see their paper for an interpretation of such information theoretic applications in terms of MARL (minimum average risk linear) predictors and filters.
### Table 7.4
Specifications of policy targets ($Y_i$).

<table>
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<tr>
<th>Specification</th>
<th>Components of $Y_i$</th>
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<tbody>
<tr>
<td>I</td>
<td>1. Per capita GNP</td>
</tr>
<tr>
<td></td>
<td>II</td>
</tr>
<tr>
<td></td>
<td>1. Per capita GNP (deflated)</td>
</tr>
<tr>
<td></td>
<td>2. Consumer price index</td>
</tr>
<tr>
<td></td>
<td>3. Unemployment rate</td>
</tr>
<tr>
<td>III</td>
<td>1-1</td>
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</tbody>
</table>

*Note:* Data are quarterly proportionate rates of change of seasonally adjusted quantities. All items in specification III are per capita nominal quantities. Data span the period 1970(I)-1978(IV).

### Table 7.5
Components of monetary aggregates.

<table>
<thead>
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<th>Symbol</th>
<th>Components</th>
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<tr>
<td>$M_f^i$ ($i=2,3$)</td>
<td>Current $M_f$</td>
</tr>
<tr>
<td>$M_p^i$ ($i=2,3$)</td>
<td>Proposed $M_f$</td>
</tr>
<tr>
<td>$M_{lp}$</td>
<td>Proposed $M_f$ less large time</td>
</tr>
<tr>
<td>$M_{mp}$</td>
<td>Current $M_f$ plus nonbank public holdings of Eurodollars, money market mutual fund shares, short-term Treasury securities, municipal bonds, RPs, and commercial paper.</td>
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</tbody>
</table>

*Note: The* aggregates are computed as proportionate rates of change in per capita seasonally adjusted nominal quantities. See Bamett, Beck, Ettin, Kalchbrenner, Lindsey, Simpson, and Tinsley (1979) for the details of the current and proposed aggregates listed in this table.

### Table 7.6
Sample estimates of percent information gain from summation to Divisia aggregation of monetary assets.

<table>
<thead>
<tr>
<th>Specification of $Y_i$</th>
<th>I</th>
<th>II</th>
<th>III</th>
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<tr>
<td>$M_2^2$</td>
<td>36.2</td>
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<td>$M_2^p$</td>
<td>16.1</td>
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<td>$M_3^p$</td>
<td>118.9</td>
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<td>92.8</td>
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<td>$M_{1b}$</td>
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<tr>
<td>$M_{mp}$</td>
<td>18.8</td>
<td>-1.0</td>
<td>66.9</td>
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The discussion in this chapter has related to the economic theories of aggregation and index numbers. However, there is also a statistical theory of index numbers which does not depend upon economic theory for its foundations. Statistical index number theory considers the ability of index numbers to pass certain classical tests, such as factor reversal and circularity tests. During the past decade, results from both approaches have converged on the Tornquist-Theil Divisia and Fisher Ideal indices as being clearly among the best, and advocates of both the economic and statistical approaches view the simple sum index as being among the very worst index numbers ever devised.12

According to Fisher, the two worst statistical properties that an index number can possess are called “bias” and “freakishness”. Regarding the simple sum (or equivalently the arithmetic average) index, which Fisher called his formula 1, Fisher (1922, p. 363) observed that, “There are two objections to Formula 1, the simple arithmetic, viz.: (1) that it is ‘simple,’ and (2) that it is arithmetic! – that it is at once freakish and biased. In the case of Sauerbeck’s index number, for instance, the bias alone reaches 36 percent!” In our case we found that much of the component information is lost unnecessarily when the components are aggregated by simple summation, and the simple sum index dismally failed to internalize the long-run substitution effects that have occurred within the money markets during the past decade. In addition, the economic restrictions on the aggregator functions necessary for simple sum aggregation were strongly rejected.13 Fisher deduced correctly (1922, p. 361) that: “The simple arithmetic (Formula 1) should not be used under any circumstances.”

We conclude with the following quotation from Fisher’s (1922, p. 29) classical book, written over half a century ago:

The simple arithmetic average is put first merely because it naturally comes first to the reader’s mind, being the most common form of average. In fields other than index numbers it is often the best form of average to use. But we shall see that the simple arithmetic average produces one of the very worst of index numbers, and if this book has no other effect than to lead to the total abandonment of the simple arithmetic type of index number, it will have served a useful purpose.

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12 As we have observed, Hulten’s and Diewert’s work strongly supports the economic foundations of the Tornquist-Theil Divisia and Fisher Ideal indices. In addition, Fisher (1922) and Theil (1967) strongly support those same indices on the basis of their statistical index number properties.

13 For further discussion of the failure of summation aggregation of monetary assets, see Bamett (1980b) and Bamett, Offenbacher, and Spindt (1981).