Bifurcation of Macroeconometric Models and Robustness of Dynamical Inferences

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Abstract: In systems theory, it is well known that the parameter spaces of dynamical systems are stratified into bifurcation regions, with each supporting a different dynamical solution regime. Some can be stable, with different characteristics, such as monotonic stability, periodic damped stability, or multiperiodic damped stability, and some can be unstable, with different characteristics, such as periodic, multiperiodic, or chaotic unstable dynamics. But in general the existence of bifurcation boundaries is normal and should be expected from most dynamical systems, whether linear or nonlinear. Bifurcation boundaries in parameter space are not evidence of model defect. While existence of such bifurcation boundaries is well known in economic theory, econometricians using macroeconometric models rarely take bifurcation into consideration when producing policy simulations from macroeconometrics models. Such models are routinely simulated only at the point estimates of the models’ parameters.

Barnett and He (1999) explored bifurcation stratification of Bergstrom and Wymer’s (1976) continuous time UK macroeconometric model. Bifurcation boundaries intersected the confidence region of the model’s parameter estimates. Since then, Barnett and his coauthors have been conducting similar studies of many other newer macroeconometric models spanning all basic categories of those models. So far, they have not found a single case in which the model’s parameter space was not subject to bifurcation stratification. In most cases, the confidence region of the parameter estimates were intersected by some of those bifurcation boundaries. The most fundamental implication of this research is that policy simulations with macroeconometric models should be conducted at multiple settings of the parameters within the confidence region. While this result would be as expected by systems theorists, the result contradicts the normal procedure in macroeconometrics of conducting policy simulations solely at the point estimates of the parameters.

This survey provides an overview of the classes of macroeconometric models for which these experiments have so far been run and emphasizes the implications for lack of robustness of conventional dynamical inferences from macroeconometric policy simulations. By making this detailed survey of past bifurcation experiments available, we hope to encourage and facilitate further research on this problem with other models and to emphasize the need for simulations at various points within the confidence region of macroeconometric models, rather than at only point estimates.
1. Bifurcation Of Macroeconomic Models

1.1. Introduction

Bifurcation has long been a topic of interest in dynamical macroeconomic systems. Bifurcation analysis is important in understanding dynamic properties of macroeconomic models as well as in selection of stabilization policies. The goal of this survey is to summarize work by William A Barnett and his coauthors on bifurcation analysis in macroeconomic models to facilitate and motivate work by others on further models. In Section 1, we introduce the concept of bifurcation and its role in studies of macroeconomic systems. We also discuss several types of bifurcations by providing examples. In Sections 2-8, we discuss bifurcation analysis and approaches with models from Barnett’s papers.

To explain what bifurcation is, we begin with a general form of many existing macroeconomic models:

\[
\mathbf{D} \mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{\theta}),
\]

(1.1)

where \( \mathbf{D} \) is the vector-valued differentiation operator, \( \mathbf{x} \) is the state vector, \( \mathbf{\theta} \) is the parameter vector, and \( \mathbf{f} \) is the vector of functions governing the dynamics of the system, with each component assumed to be smooth in a local region of interest.

In system (1.1), we are particularly interested in the settings of the parameter vector, \( \mathbf{\theta} \). Assume \( \mathbf{\theta} \) takes values within a theoretically feasible set \( \Theta \). The value of \( \mathbf{\theta} \) can affect the dynamics of the system in very fundamental ways. If a small change in the parameters can fundamentally alter the nature of dynamical solution path, we say a bifurcation occurs in the system through the change of parameters.

Bifurcation refers to a change in qualitative features, rather than quantitative features, of the solution dynamics as parameter values change. A change in quantitative features of dynamical solutions may refer to a change in such properties as the period or amplitude of cycles, as parameters change, while a change in qualitative features of may refer to such changes as changes from stable monotonic convergence to stable damped convergence, or changes from stability to periodic instability, or changes from periodic to multiperiodic instability, or changes to chaotic instability.

A point within the parameter space at which a change in a quality of the dynamical solution path occurs defines a point on a bifurcation boundary. At the bifurcation point, the structure of the dynamic system may change its stability and split into new structures, producing very complex behavior. Fundamental different dynamical solution properties can occur when parameters are close together but on different sides of a bifurcation boundary.
parameter set can be stratified into several subsets with different types of dynamics within each subset.

There are several types of bifurcation boundaries, such as Hopf, pitchfork, saddle-node, transcritical, and singularity bifurcation. Each type of bifurcation produces a different type of qualitative dynamic change. We illustrate these different types of bifurcation by providing examples in section 1.3. Bifurcation boundaries have been discovered in many macroeconomic systems. For example, Hopf bifurcations have been found in growth models (e.g., Benhabib and Nishimura (1979), Boldrin and Woodford (1990), Dockner and Feichtinger (1991), and Nishimura and Takahashi (1992)) and in overlapping generations models (Barnett and Duzhak (2010), page 3). Pitchfork bifurcations have been found in the tatonnement process (e.g., Bala (1997) and Scarf (1960)). Transcritical bifurcations have been found in Bergstrom and Wymer’s (1976) UK model (Barnett and He (1999)) and singularity bifurcation in Leeper and Sims’ Euler-equation model (Barnett and He (2008)).

One reason for concern about bifurcation in macroeconomic models is because changes in parameters could affect dynamic behaviors of the models and consequently the outcomes of imposition of policy rules. For example, Bergstrom and Wymer’s (1976) UK model operates close to bifurcation boundaries between stable and unstable regions of the parameter space. In this case, if a bifurcation boundary intersects the confidence region of the parameter estimates, different qualitative properties of solution can exist within this confidence region. As a result, robustness of inferences about dynamics can be damaged, especially if inferences about dynamics are based on model simulations with the parameters set only at their point estimates. When confidence regions are stratified by bifurcation boundaries, dynamical inferences need to be based on simulations at points within each of the stratified subsets of the confidence region.

Knowledge of bifurcation boundaries is directly useful in policy selection. If the system is unstable, a successful policy would bifurcate the system from the unstable to stable region. In that sense, stabilization policy can be viewed as bifurcation selection. As illustrated in section 2, Barnett and He (2002) have shown that successful bifurcation policy selection can be difficult to design.

Barnett’s work has found bifurcation phenomena in every macroeconomic model that he and his coauthors have so far explored. Barnett and He (1999,2002) examined the dynamics of Bergstrom-Wymer’s continuous-time dynamic macroeconomic model of the UK economy and found both transcritical and Hopf bifurcation boundaries. Barnett and He (2008) estimated and displayed singularity bifurcation boundaries for the Leeper and Sims (1994) Euler equations model. Barnett and Duzhak (2010) found Hopf and period doubling bifurcations in a New Keynesian model. Banerjee, Barnett, Duzhak and Gopalan (2011) examined the possibility of

This survey is organized in the chronological order of Barnett’s work on bifurcation of macroeconomic models, from early models to many of the most recent models.

### 1.2. Stability

There are two possible approaches to analyze bifurcation phenomena: global and local. Methods in Barnett’s current papers have used local analysis, which is analysis of the linearized dynamic system in a neighborhood of the steady state. In his papers, equation (1.1) is linearized in the form

\[
\mathbf{Dx} = \mathbf{A}(\theta)\mathbf{x} + \mathbf{F}(\mathbf{x}, \theta),
\]

where \(\mathbf{A}(\theta)\) is the Jacobian matrix of \(\mathbf{f}(\mathbf{x}, \theta)\), and \(\mathbf{F}(\mathbf{x}, \theta) = \mathbf{f}(\mathbf{x}, \theta) - \mathbf{A}(\theta)\mathbf{x} = o(\mathbf{x}, \theta)\) is the vector of higher order term. Define \(\mathbf{x}^*\) to be the system’s steady state equilibrium, such that \(\mathbf{f}(\mathbf{x}^*, \theta) = 0\), and redefine the variables such that the steady state is the point \(\mathbf{x}^* = 0\) by replacing \(\mathbf{x}\) with \(\mathbf{x} - \mathbf{x}^*\).

The local stability of (1.1), for small perturbation away from the equilibrium, can be studied through the eigenvalues of \(\mathbf{A}(\theta)\), which is a matrix-valued function of the parameters \(\theta\), as follows:

(a) If all eigenvalues of \(\mathbf{A}(\theta)\) have strictly negative real parts, then (1.1) is locally asymptotically stable in the neighborhood of \(\mathbf{x} = 0\).

(b) If at least one of the eigenvalues of \(\mathbf{A}(\theta)\) has positive real part, then (1.1) is locally asymptotically unstable in the neighborhood of \(\mathbf{x} = 0\).

(c) If all eigenvalues of \(\mathbf{A}(\theta)\) have nonpositive real parts and at least one has zero real part, the stability of (1.1) usually cannot be determined from the matrix \(\mathbf{A}(\theta)\). Higher order terms need to be analyzed to determine the stability of the system.

For far more complicated global analysis, higher order terms must be considered, since the perturbations away from the equilibrium can be large. Analysis of \(\mathbf{A}(\theta)\) alone may not be useful. More research on global analysis of macroeconomic models is needed.

It is important to know at what parameter values, \(\theta\), the system (1.1) is stable. But it is also important to know the nature of the instability, when the system is unstable, such as periodic, multiperiodic, or chaotic. It also is important to know the nature of the stability, when the system is stable, such as monotonically convergent, damped single-periodic convergent, or damped multiperiodic convergent.
To analyze the stability properties of the system, the bifurcation boundaries must be found. As evident from (a)-(c), the boundary could only be found under (c), for example, if \( \mathbf{A}(\theta) \) has at least one zero eigenvalue. In that case,

\[
\text{det} (\mathbf{A}(\theta)) = 0. \tag{1.3}
\]

In some cases in which (1.3) identifies the stability boundary, that boundary can be hard to locate. In Barnett and He (1999, 2000), various methods are applied to locate the bifurcation boundaries characterized by (1.3). Equation (1.3) can be difficult to solve in closed form when \( \theta \) is multi-dimensional. As a result, numerical methods are extensively used for solving (1.3).

Before proceeding to the next section, we introduce the definition of hyperbolic.

**Definition 1.1.** An equilibrium point \( \mathbf{x}^* \) of (1) is called hyperbolic, if the coefficient matrix \( \mathbf{A}(\theta) \) has no eigenvalues with zero real parts.

The asymptotical behavior of (1.1) at a hyperbolic point is determined by criteria (a)-(b) above. There are structural changes at a sufficiently small neighborhood of a hyperbolic equilibrium. We are more interested in the behavior at non-hyperbolic equilibria, since bifurcations can occur in a local neighborhood of non-hyperbolic equilibria.

In local bifurcation analysis, there are the following three codimension-1 types of bifurcation. They violate the hyperbolicity condition.

**Definition 1.2.** Bifurcation associated with the appearance of an eigenvalue \( \lambda_i = 1 \) is called a fold (or saddle-node) bifurcation.

**Definition 1.3.** Bifurcation associated with the appearance of an eigenvalue \( \lambda_i = -1 \) is called flip (or period-doubling) bifurcation.

**Definition 1.4.** Bifurcation associated with the presence of a pair of complex conjugate eigenvalues, \( \lambda_1 = e^{i\theta_0} \) and \( \lambda_1 = e^{-i\theta_0} \), for \( 0 < \theta_0 < \pi \), is called a Hopf bifurcation.

However, nonhyperbolicity doesn’t necessarily induce bifurcations. In addition to hyperbolicity, derivative conditions are important in determining whether there is a bifurcation. An important theorem on Hopf bifurcation will be introduce in section 1.3.4.

### 1.3. Types of Bifurcations

#### 1.3.1. Transcritical Bifurcations

Given by the Stomayor’s theorem in Sotomayor (1973), a transcritical bifurcation occurs when a system has a non-hyperbolic equilibrium with a geometrically simple zero eigenvalue at the bifurcation point and when additional transversality conditions are satisfied.
For a one-dimensional system,

\[ Dx = G(x, \theta), \]

the transversality conditions for a transcritical bifurcation at \((x, \theta) = (0,0)\) are

\[ G(0,0) = G_x(0,0) = G_\theta(0,0) = 0, \quad G_{xx}(0,0) \neq 0, \quad \text{and} \quad G_{\theta x}^2 - G_{xx}G_{\theta \theta}(0,0) > 0. \]  

(1.4)

An example of such a form is

\[ Dx = \theta x - x^2. \]  

(1.5)

To find its equilibria, set \(Dx = 0\) and see from (1.5) that the steady state equilibria of the system are at \(x^* = 0\) and \(x^* = \theta\). It follows that system (1.5) is stable around the equilibrium \(x^* = 0\) for \(\theta < 0\), and unstable for \(\theta > 0\). System (1.5) is stable around the equilibrium \(x^* = \theta\) for \(\theta > 0\), and unstable for \(\theta < 0\). The nature of the dynamics changes as the system bifurcates at the origin.

Transcritical bifurcations have been found in high-dimensional continuous-time macroeconomic systems, but in high dimensional cases, transversality conditions have to be verified on a manifold. Details are provided in Guckenheimer and Holmes (1983).

1.3.2. Pitchfork Bifurcations

Consider a one-variable, one-parameter differential equation

\[ Dx = f(x, \theta). \]

Suppose that there exists an equilibrium \(x^*\) and a parameter value \(\theta^*\) such that \((x^*, \theta^*)\) satisfies the following conditions:

\[ (a) \quad \frac{\partial f(x, \theta^*)}{\partial x}|_{x=x^*} = 0, \]

\[ (b) \quad \frac{\partial^3 f(x, \theta^*)}{\partial x^3}|_{x=x^*} \neq 0, \]

\[ (c) \quad \frac{\partial^2 f(x, \theta)}{\partial x \partial \theta}|_{x=x^*, \theta=\theta^*} \neq 0. \]

Then \((x^*, \theta^*)\) is a pitchfork bifurcation point. When the parameter \(\theta\) crosses \(\theta^*\), the dynamic system could change from stable to unstable, depending on the signs of the transversality condition.

An example of such form is
\[ Dx = \theta x - x^3. \]

To find its equilibria, set \( Dx = 0 \) and find that the steady state equilibria of the system are at \( x^* = 0 \) and \( x^* = \pm \sqrt{\theta} \). When \( x = 0 \) and \( \theta = 0 \), \( \frac{\partial f(x, \theta)}{\partial x} = \theta - 3x^2 = 0 \), and all the transversality conditions are satisfied at this point. It follows that the system is stable when \( \theta < 0 \) at the equilibrium \( x^* = 0 \), and unstable at this point when \( \theta > 0 \). The two other equilibria \( x^* = \pm \sqrt{\theta} \) are stable for \( \theta > 0 \). Pitchfork bifurcation is said to be supercritical in this case. Otherwise, it is said to be subcritical.

There is an example from Bala (1997) which explains how pitchfork bifurcation can occur in the tatonnement process. Suppose the economy consists of two goods and two agents. The agents have CES utility functions parameterized by \( \mu \in [0,1] \). The utility functions and endowments of agents 1 and 2 are

\[
\begin{align*}
\mu^1(x_1, x_2, \mu) &= -x_1^{\mu-1} - 2x_1^{\mu-1}x_2^{\mu-1}, \\
\mu^2(x_1, x_2, \mu) &= -x_2^{\mu-1} - 2x_1^{\mu-1}x_2^{\mu-1}.
\end{align*}
\]

The tatonnement process for the economy is given by

\[ Dp = Z_1(p, \mu). \]

Bala (1997) shows that pitchfork bifurcation exists in this system and that for any \( \mu \in \left(\frac{3}{4}, 1\right) \) the economy has three equilibria.

**1.3.3. Saddle-Node Bifurcations**

For a general one-dimensional system,

\[ Dx = f(x, \theta). \]

A saddle-node point \((x^*, \theta^*)\) satisfies the equilibrium condition \( f(x^*, \theta^*) = 0 \) and the Jacobian condition \( \frac{\partial f(x, \theta^*)}{\partial x}\bigg|_{x=x^*} = 0 \), as well as the transversality conditions for bifurcation

\[
\begin{align*}
(a) & \quad \frac{\partial f(x, \theta)}{\partial \theta}\bigg|_{x=x^*, \theta=\theta^*} \neq 0, \\
(b) & \quad \frac{\partial^2 f(x, \theta)}{\partial x^2}\bigg|_{x=x^*, \theta=\theta^*} \neq 0.
\end{align*}
\]

Sotomayor (1973) shows that transversality conditions for high-dimensional systems can also be formulated.
A simple system with a saddle-node bifurcation is

\[ Dx = \theta - x^2. \]

To find its equilibria, set \( Dx = 0 \) and find that the equilibria are \( x^* = \pm \sqrt{\theta} \), which requires \( \theta \) to be nonnegative. Therefore, there exist no equilibria for \( \theta < 0 \), and there exist two equilibria at \( x^* = \pm \sqrt{\theta} \) when \( \theta > 0 \). It follows that when \( \theta > 0 \), the system is stable at \( x^* = \sqrt{\theta} \) and unstable at \( x^* = -\sqrt{\theta} \). In this example, bifurcation occurs at the origin, which is called the saddle node, causing transition between stability and instability along the path of equilibria to the right of the origin.

An example in Barnett and He (2004) is a model in Grandolfo (1996), using the following economic system

\[ Dv = v[F(r, \alpha) - S(r)], \]

which exhibits a saddle-node bifurcation. Here \( r \) is the spot exchange rate defined as domestic currency per foreign currency, \( v > 0 \) is the adjustment speed, \( \alpha \) is a parameter, and \( \frac{\partial F}{\partial \alpha} > 0 \).

The differential equation indicates that the exchange rate adjusts according to excess demand. In deriving the model, an assumption is that the demand for and supply of foreign exchange come solely from traders, and the supply curve is backward-bending. Therefore, the demand curve and supply curve could have two intersection points and one point of tangency between the two curves. It can be verified that the transversality conditions for saddle-node bifurcations are satisfied. Hence the tangent point \((r^*, \alpha^*)\) is a saddle-node bifurcation.

1.3.4. Hopf Bifurcations

Hopf bifurcation occurs at the points at which the system has a non-hyperbolic equilibrium with a pair of purely imaginary eigenvalues, but without zero eigenvalues. Regarding the additional eigenvalue and transversality conditions that must be satisfied for Hopf bifurcation, see the Hopf Theorem in Guckenheimer and Holmes (1983). A lengthy account of transversality conditions are given in Glendinning (1994). The basic requirements of the presence of Hopf bifurcation are (1) the occurrence of a pair of purely imaginary eigenvalues (hence the dimension of a system needs to be at least two) and (2) the system crosses the stability boundary with nonzero speed. In the special case of \( n=2 \), the following theorem is based upon the Hopf Bifurcation Theorem in Gandolfo (2010, ch. 24, p.497).

**Theorem 1.1.** (Existence of Hopf Bifurcation in 2 dimensions) Consider the two-dimensional non-linear difference system with one parameter

\[ y_{t+1} = \Phi(y_t, \alpha), \]
and suppose that for each $\alpha$ in the relevant interval there exists a smooth family of equilibrium points, $y_e = y_e(\alpha)$, at which the eigenvalues are complex conjugates, $\lambda_{1,2} = \theta(\alpha) + i\omega(\alpha)$. If there is a critical value $\alpha_0$ of the parameter such that

a. the eigenvalues’ modulus becomes unity at $\alpha_0$, but the eigenvalues are not roots of unity (from the first up to the fourth), namely

$$|\lambda_{1,2}(\alpha_0)| = \sqrt{\theta^2 + \omega^2} = 1, \quad \lambda_{1,2}^j(\alpha_0) \neq 1 \text{ for } j = 1, 2, 3, 4,$$

and

b. $\frac{d|\lambda_{1,2}(\alpha)|}{d\alpha} \bigg|_{\alpha=\alpha_0} \neq 0$,

then there is an invariant closed curve bifurcating from $\alpha_0$.

This theorem only applies with a $2 \times 2$ Jacobian. The more general case requires the rest of the eigenvalues to have a real part less than zero. This theorem is widely applied to find the criterion for the presence of a Hopf bifurcation.

An example in the two-dimensional system is

$$Dx = -y + x(\theta - (x^2 + y^2)),$$

$$Dy = x + y(\theta - (x^2 + y^2)).$$

To find the equilibria, set $Dx = Dy = 0$. One equilibrium is $x^* = y^* = 0$ with stability occurring for $\theta < 0$ and the instability occurring for $\theta > 0$. That equilibrium has a pair of conjugate eigenvalues $\theta + i$ and $\theta - i$. The eigenvalues become purely imaginary, when $\theta = 0$, which is the bifurcation point.

To find Hopf bifurcation boundaries, let $p(s) = det(sI - A)$ be the characteristic polynomial of $A$, and write it as

$$p(s) = c_0 + c_1s + c_2s^2 + c_3s^3 + \cdots + c_{n-1}s^{n-1} + s^n.$$

Construct the following $(n - 1)$ by $(n - 1)$ matrix
Let $S_0$ be obtained by deleting rows 1 and $\frac{n}{2}$ and columns 1 and 2, and let $S_1$ be obtained by deleting rows 1 and $\frac{n}{2}$ and column 1 and 3. The matrix $A(\theta)$ has exactly one pair of purely imaginary eigenvalues (Guckenheimer et al. (1997)) if

$$\det(S) = 0, \quad \det(S_0) \det(S_1) > 0.$$  \hfill (1.6)

If $\det(S) \neq 0$ or if $\det(S_0) \det(S_1) < 0$, then $A(\theta)$ has no purely imaginary eigenvalues. If $\det(S) = 0$ and $\det(S_0) \det(S_1) = 0$, then $A(\theta)$ may have more than one pair of purely imaginary eigenvalues. If $A(\theta)$ has exactly one pair of purely imaginary eigenvalues and $\det(A(\theta)) \neq 0$, and if some additional transversality conditions hold, this point is on a Hopf bifurcation boundary. The following condition can be used to find candidates for bifurcation boundaries:

$$\det(S) = 0, \quad \det(S_0) \det(S_1) \geq 0.$$  \hfill (1.7)

Solving (1.7) is rarely analytically possible. However, the following numerical procedure provided in Barnett and He (1999) can be applied to find bifurcation boundaries. For the sake of simplicity, they only consider two parameters $\theta_1$ and $\theta_2$.

**Procedure (P1)**

1. For any fixed $\theta_1$, treat $\theta_2$ as a function of $\theta_1$ and find the value of $\theta_2$ satisfying the condition $h(\theta_2) = \det(A(\theta)) = 0$. First find the number of zeros of $h(\theta_2)$. Starting with approximations of zeros, use the following gradient algorithm to find all zeros of $h(\theta_2)$:

$$\theta_2(n + 1) = \theta_2(n) - a_n h(\theta_2) \bigg|_{\theta_2=\theta_2(n)}$$  \hfill (1.8)

where $\{a_n, n = 0,1,2 \ldots \}$ is a sequence of positive step sizes.

2. Repeat the same procedure to find all $\theta_2$ satisfying (1.7).

3. Plot all the pairs $(\theta_1, \theta_2)$.
(4) Check all parts of the plot to find the segments representing the bifurcation boundaries. Then parts of the curve found in (1) are boundaries of saddle-node bifurcations, while parts of the curve found in (2) are boundaries of Hopf bifurcations, if the required transversality conditions are satisfied.

Case studies using Procedure (P1) will be shown in section 2, when we survey research on bifurcation phenomena in the continuous-time UK model.

Hopf bifurcation is the most studied type of bifurcation in economics. The first theoretical work on Hopf bifurcation was conducted by Poincaré (1892). Andronov (1929) and his coauthors developed important tools for analyzing nonlinear dynamical system, formulating a theorem on Hopf bifurcation for the first time. Bifurcation that results from parameters crossing a bifurcation boundary such that the solutions change from damped stable to unstable limit cycle is called Poincaré-Andronov-Hopf bifurcation. Both Poincaré and Andronov’s work was concerned with two-dimensional vector fields. Later, a general theorem on the existence of Hopf bifurcation was proved by Hopf (1942). That theorem is valid in $n$ dimensions.

Torre (1977) and Benhabib and Nishimura (1979) were among the first studies on Hopf bifurcation in the field of economics. Torre studied Keynesian systems and found the appearance of a limit cycle associated with a Hopf bifurcation boundary. Benhabib and Nishimura analyzed a multi-sector neoclassical optimal growth model and showed that a closed invariant curve might emerge as the result of optimization. Historically, optimal growth theory has received the most attention in bifurcation analysis. Hopf bifurcations have also been found in overlapping generations models. These studies illustrate that the existence of a Hopf bifurcation boundary in an economic model results in a solution following closed curves around the stationary state, with the solution paths being stable or unstable, depending upon which side of the bifurcation boundary contains the parameter values. More recent studies finding Hopf bifurcation in econometric models include Barnett and He (1999, 2002, 2004, 2006, 2008), who found bifurcation boundaries within the parameter spaces of the Bergstrom continuous-time model of the UK economy and the Leeper and Sims Euler-equations model of the United States economy.

In the literature on chaos, Hopf bifurcation is fundamental, since the first bifurcation along the route to chaos is the loss of stability to a simple single-periodic limit cycle, as produced by Hopf bifurcation. As a result, Hopf bifurcation boundaries tend to be encountered as boundaries between stability and instability, rather than between two forms of stability or between two forms of instability.

1.3.5. Singularity-Induced Bifurcations
This section is devoted to a dramatic kind of bifurcation found by Barnett and He (2008) in the Leeper and Sims (1977) model—singularity-induced bifurcation. Some macroeconomic models, such as the dynamic Leontief model (Luenberger and Arbel (1977)) and the Leeper and Sims (1994) model, have the form

$$\mathbf{B} \mathbf{x}(t + 1) = \mathbf{A} \mathbf{x}(t) + \mathbf{f}(t).$$

(1.9)

Here $\mathbf{x}(t)$ is the state vector, $\mathbf{f}(t)$ is the vector of driving variables, $t$ is time, and $\mathbf{B}$ and $\mathbf{A}$ are constant matrices of appropriate dimensions. If $\mathbf{f}(t) = \mathbf{0}$, the system (1.9) is in the class of autonomous systems. Barnett and He (2008) illustrate the autonomous cases of (1.9).

If $\mathbf{B}$ is invertible, then (1.9) will be consistent with the discrete-time form of the system (1.1), as is shown by inverting $\mathbf{B}$ to acquire:

$$\mathbf{x}(t + 1) = \mathbf{B}^{-1} \mathbf{A} \mathbf{x}(t) + \mathbf{B}^{-1} \mathbf{f}(t),$$

so that

$$\mathbf{x}(t + 1) - \mathbf{x}(t) = \mathbf{B}^{-1} \mathbf{A} \mathbf{x}(t) - \mathbf{x}(t) + \mathbf{B}^{-1} \mathbf{f}(t) = (\mathbf{B}^{-1} \mathbf{A} - \mathbf{I}) \mathbf{x}(t) + \mathbf{B}^{-1} \mathbf{f}(t),$$

which is in the form of (1.1).

Barnett and He (2008) are more interested in the case in which the matrix $\mathbf{B}$ is singular. They rewrite (1.9) by generalizing to permit nonlinearity as follows:

$$\mathbf{B}(\mathbf{x}(t), \theta) \mathbf{D} \mathbf{x} = \mathbf{F}(\mathbf{x}(t), \mathbf{f}(t), \theta).$$

(1.10)

Here $\mathbf{f}(t)$ is the vector of driving variables and $t$ is time. Consider the autonomous cases in which $\mathbf{f}(t) = \mathbf{0}$. Singularity-induced bifurcation occurs when the rank of $\mathbf{B}(\mathbf{x}, \theta)$ changes, as from an invertible matrix to a singular one. For such changes to occur, the matrix must depend on $\theta$. In such cases, the dimension of the dynamical part of the system changes accordingly.

The dependency of the matrix $\mathbf{B}(\mathbf{x}, \theta)$ upon $\theta$ can be through any form of point-to-matrix mapping producing a dependence of $\mathbf{B}(\mathbf{x}, \theta)$ upon $\theta$. There need not exist a closed form algebraic dependence of the elements of $\mathbf{B}(\mathbf{x}, \theta)$ upon $\theta$. If $\mathbf{B}(\mathbf{x}, \theta)$ does not depend on $\theta$, then singularity of $\mathbf{B}(\mathbf{x}, \theta)$ is not sufficient for (1.10) to be able to produce singularity bifurcation, since the rank of $\mathbf{B}(\mathbf{x}, \theta)$ will not change as $\theta$ changes. For example, the Leontief model described by Luenberger and Arbel (1977) is in the class of systems (1.9) with a singular matrix $\mathbf{B}$, but no singularity bifurcation boundary has been found within that model. When $\mathbf{B} = \mathbf{I}$, system (1.10) becomes system (1.1).
In general, the structural properties of the dynamical implicit function system (1.10) can be substantially more complex than those for the closed form system (1.1). When \( \mathbf{B} \neq \mathbf{I} \), the matrix \( \mathbf{B} \) can take values producing a large number of dynamical possibilities for (1.10).

The systems (1.9) and (1.10) are often referred to as differential-algebraic systems. Consider the two-dimensional state-space case, with \( \mathbf{x} = (x_1, x_2) \). An appropriate coordinate transformation allows (1.10) to become the following form:

\[
\begin{align*}
\mathbf{B}_1(x_1, x_2, \theta) \mathbf{D}x_1 &= F_1(x_1, x_2, \theta), \\
0 &= F_2(x_1, x_2, \theta).
\end{align*}
\]

Barnett and He (2008) provide four examples to demonstrate the complexity of bifurcation behaviors that can be produced from system (1.10). The first two examples, in which \( \mathbf{B} \) does not depend on \( \theta \), do not produce singularity bifurcations. In the second two examples, \( \mathbf{B} \) does depend on \( \theta \). In those two examples, Barnett and He (2008) find singularity bifurcation regions within the parameter spaces.

**Example 1.** Consider the following system modified from system (1.5), which can produce transcritical bifurcation:

\[
\begin{align*}
Dx &= \theta x - x^2, \quad \text{(1.11)} \\
0 &= x - y^2. \quad \text{(1.12)}
\end{align*}
\]

Comparing with the general form of (1.10), observe that

\[
\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]

which is singular but does not depend upon the value of \( \theta \). By setting \( Dx = 0 \), find equilibria \( (x^*, y^*) = (0,0) \) and \( (\theta, \pm \sqrt{\theta}) \). Near the equilibrium \( (x^*, y^*) = (0,0) \), the system (1.11)-(1.12) is stable for \( \theta < 0 \) and unstable for \( \theta > 0 \). The equilibria \( (x^*, y^*) = (\theta, \pm \sqrt{\theta}) \) are undefined, when \( \theta < 0 \), and stable, when \( \theta > 0 \).

The bifurcation point is \( (x, y, \theta) = (0,0,0) \). The movement from the stable equilibria at \( (x^*, y^*) = (0,0) \) with negative \( \theta \) to the unstable equilibria at the same point \( (x^*, y^*) = (0,0) \) with positive \( \theta \) will cause bifurcation from stability to instability. It is also possible to bifurcate from the stable equilibrium at \( (x^*, y^*) = (0,0) \) with negative \( \theta \) to the stable equilibria along the three dimensional parabola \( \{(x, y, \theta): x = \theta, y = \pm \sqrt{\theta}, \theta > 0\} \). In this case, even if the dynamics remain stable before and after the bifurcation, the bifurcation changes the nature of the dynamics in some ways. If the confidence region contains the point \( (x^*, y^*, \theta) = (0,0,0) \),
three kinds of equilibria are possible within the confidence region: one unstable and two stable. Different forms of disequilibrium dynamics are likely to exist around each.

Although $B$ is singular, the bifurcation point does not produce singularity bifurcation, since $B$ does not depend upon $\theta$. Notice before and after bifurcation, the number of differential equations and the number of algebraic equations remain unchanged. However, singularity bifurcation causes change in the dimension of the dynamics.

**Example 2.** Consider the following system modified from system (1.7), which can produce saddle-node bifurcation:

\[
\begin{align*}
D_{xx} &= \theta - x^2, \quad (1.13) \\
0 &= x - y^2. \quad (1.14)
\end{align*}
\]

Again, comparing with the general form of (1.10), Barnett and He (2008) observe that

\[
B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]

which is singular but does not depend upon the value of $\theta$.

To find the equilibria, set $D_{xx} = 0$ and find the equilibria are at $(x^*, y^*) = (\sqrt{\theta}, \pm \sqrt{\theta})$, defined only for $\theta \geq 0$. The system ((1.13),(1.14)) is stable around both of the equilibria, $(x^*, y^*) = (\sqrt{\theta}, \pm \sqrt{\theta})$ and $(x^*, y^*) = (\sqrt{\theta}, \pm i \sqrt{\theta})$. The bifurcation point between the two stable regions is $(x^*, y^*, \theta) = (0,0,0)$. For the same reason as in the previous example, the bifurcation point does not produce singularity bifurcation. There is no change in the dimension at the origin. However, a true singularity bifurcation would result in a change in the mix of algebraic and differential equations and dramatic change in the dimension of the state space dynamics. The following two examples illustrate such cases.

**Example 3.** Consider the following system:

\[
\begin{align*}
D_{xx} &= ax - x^2, \text{ with } a > 0, \quad (1.15) \\
\theta D_{yy} &= x - y^2. \quad (1.16)
\end{align*}
\]

Barnett and He (2008) observe that

\[
B = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix},
\]
which does depend upon the parameter $\theta$. Notice when $\theta = 0$, the system has one differential equation (1.15) and one algebraic equation (1.16). If $\theta \neq 0$, the system has two differential equations (1.15) and (1.16) with no algebraic equations for nonzero $\theta$.

To find the equilibria, set $Dx = Dy = 0$ and find that the equilibria are $(x^*, y^*) = (0,0)$ and $(a, \pm \sqrt{a})$. For any value of $\theta$, system ((1.15),(1.16)) is unstable around the equilibrium at $(x^*, y^*) = (0,0)$. The equilibrium $(x^*, y^*) = (a, \sqrt{a})$ is unstable for $\theta < 0$ and stable for $\theta > 0$. The equilibrium $(x^*, y^*) = (a, -\sqrt{a})$ is unstable for $\theta > 0$ and stable for $\theta < 0$.

Without loss of generality, normalize $a$ to be 1. When $\theta = 0$, the system’s behavior degenerates into movement along the curve $x - y^2 = 0$. There are one unstable equilibrium at $(0,0)$ and the two stable equilibria at $(1,1)$ and $(1,-1)$, with the disequilibrium dynamics constrained to the path $x - y^2 = 0$. When $\theta$ is nonzero, the dynamics of the system move throughout the two-dimensional state space. The singularity bifurcation, produced by the transition from nonzero $\theta$ to zero, results in the dramatic drop in the dimension.

It is important to note the change in dynamical properties produced by singularity bifurcation, even if the bifurcation does not change between stability and instability. For example, if $\theta$ changes from positive to zero, when $(x,y)$ is at the equilibrium $(1,1)$, the system will remain stable, but disequilibrium dynamics will drop in dimension to a lower dimensional space. If $\theta$ changes from positive to zero, when $(x,y)$ is at the equilibrium $(0,0)$, the dynamics will remain unstable both before and after the bifurcation, but the dimension of the dynamics will drop. If $\theta$ changes from positive to zero, when $(x,y)$ is at the equilibrium $(1,-1)$, the dynamics will change from unstable to stable and the dimension of the dynamics will also drop. In all of these cases, the nature of the disequilibrium dynamics changes dramatically, even if there is no transition between stability and instability.

Unless economic theory provides a reason to consider the dynamics, when parameters are set directly on a bifurcation point, the change in dynamics from one side of bifurcation point to the other side is far more important than the change in dynamics from parameter settings on one side of a bifurcation point to settings directly on a bifurcation point. Bifurcation regions are measure zero subsets of the parameter space. Hence, the effect of changing the parameter between strictly negative settings of $\theta$ and strictly positive setting of $\theta$ is of particular importance. The comparison of the dynamics between two such nonzero settings does not display the dramatic drop into the “black hole” space when $\theta = 0$, but the shift between positive and negative values of $\theta$ does cause the stability and instability of the equilibria $(1,1)$ and $(1,-1)$ to be interchanged. Observing the direction of the arrows of the disequilibrium paths around the unstable equilibrium $(0,0)$ in a phase space plot, Barnett and
He (2008) observe that even in the vicinity of that always unstable equilibrium, the nature of the unstable dynamics will change substantially, when the sign of $\theta$ changes.

This observation will be important in understanding the research with the Leeper and Sims model in section 3. In that model, Barnett and He (2003) found the system is unstable on both sides of the singularity bifurcation, which is within the neighborhood of the parameter estimates. In fact the singularity bifurcation boundary intersects the confidence region around the parameter estimates.

**Example 4.** Consider the following system:

$$Dx = ax - x^2,\text{ with } a > 0,$$  \hspace{1cm} (1.17)

$$\theta Dy = x - y.$$  \hspace{1cm} (1.18)

Barnett and He (2008) observe that

$$B = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix}.$$

In this case, for every $\theta$ the equilibria are $(x^*, y^*) = (0, 0)$ and $(a, a)$. The system is unstable around the equilibrium $(x^*, y^*) = (0, 0)$ for any value of $\theta$. The equilibrium $(x^*, y^*) = (a, a)$ is unstable for $\theta < 0$ and stable for $\theta \geq 0$. When $\theta = 0$, the variable $y$ in ((1.17),(1.18)) is just a replica of the variable $x$ in system, since equation (1.18) becomes the algebraic constraint $y = x$. The disequilibrium dynamics are one-dimensional along the ray through the origin, regardless of whether stable in the vicinity of $(1, 1)$ or unstable in the neighborhood of $(0,0)$. However, when $\theta \neq 0$, the system moves into the two-dimensional space.

Observe that $(0,0)$ remains unstable and $(1, 1)$ remains stable in both cases, when $\theta \neq 0$ and $\theta = 0$. The singularity bifurcation that caused transition between the two dimensional space and the one dimensional path need not cause a change between stability and instability. Stability can remain stable, and instability can remain unstable, but with dramatic change in the nature of the dynamics. Also observe that the nature of the dynamics with $\theta$ small and positive is very different from that with $\theta$ small and negative. In particular, the equilibrium at $(x^*, y^*) = (1,1)$ is stable in the former case and unstable in the latter case. There is little robustness of dynamical inference to small changes of $\theta$ in the vicinity of the bifurcation boundary, even if the startling drop into the measure-zero “black hole” at exactly $\theta = 0$ is never encountered. On the more general subject of robustness of inference in dynamic models, see Barnett and Binner (2004, part 4).
Changes in the dynamical properties of (1.10) through singularity bifurcation can occur, even when the parameters θ do not appear directly within the matrix $B = B(x, \theta)$ itself, but rather affect $B$ through a mapping from outside $B$, as illustrated in the following example.

**Example 5.** Consider the following system:

$$
\begin{align*}
Dx_1 &= x_3, \\
Dx_2 &= -x_2, \\
0 &= x_1 + x_2 + \theta x_3,
\end{align*}
$$

which has the following singular matrix $B$:

$$
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix},
$$

where $Dx = (Dx_1, Dx_2, Dx_3)'$.

Solving $Dx = 0$, the only equilibrium is at $x^* = (x_1^*, x_2^*, x_3^*) = (0,0,0)$. For any $\theta \neq 0$, solving the last equation for $x_3$ and substituting into the first equation results in the two equation system

$$
\begin{align*}
Dx_1 &= -(x_1 + x_2)/\theta, \\
Dx_2 &= -x_2,
\end{align*}
$$

which is stable at its 2-dimensional equilibrium $x^* = (x_1^*, x_2^*) = (0,0)$ for $\theta > 0$ and unstable at that equilibrium for $\theta < 0$. Observe that the matrix $B$ now is the nonsingular matrix $B = I$.

Barnett and He (2008) want to consider what happens on the singularity bifurcation boundary with $\theta = 0$. Setting $\theta = 0$, they find that system (1.19) becomes

$$
\begin{align*}
x_1 &= -x_2, \\
Dx_2 &= -x_2, \\
x_3 &= x_2,
\end{align*}
$$

for all $t > 0$. This system has the following singular matrix $B$:

$$
B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$
Note the different order of the dynamics in (1.22) from that of (1.21). In system (1.22), there are two algebraic constraints and one differential equation, while system (1.21) has two differential equations and no algebraic constraints. Clearly the matrix $B$ is different in the two cases and the rank of $B$ has changed between the two cases. Yet one would not have anticipated this change from inspection of the general form of the system, (1.19), since its matrix $B$, (1.20), does not contain the model’s parameter within the matrix itself. In short, the matrix can depend upon the parameters, and singularity bifurcation can occur, even if there does not exist a direct closed-form algebraic representation of the dependence of $B$ upon the parameters.

Barnett and He (2003) found singularity bifurcation in their research on the Leeper and Sims Euler-equations macroeconometric model. They believe singularity bifurcations will be found to have important implications for robustness of dynamic inferences with other modern Euler-equations macroeconometric models. Euler equation systems are in implicit function form and rarely can be solved for closed form representations. The implicit function system (1.9),(1.10) can produce singularity bifurcation, while the closed form differential equation system (1.1) cannot produce singularity bifurcation. Singularity bifurcation did not appear with older algebraically-solvable macroeconometric model. Because of their implicit function form, singularity bifurcation needs to be taken seriously with modern Euler equations models.

In the unlikely case that the parameters fall exactly on the measure-zero singularity bifurcation boundary, the dynamics of the system drop into a “black hole” lower-dimensional state space. Although that dimensional collapse does not occur on either side of the boundary, the dynamical properties on one side of the boundary can be very different from those on the other side. It is important to recognize that the startling differences in dynamics on the two sides of a singularity bifurcation boundary need not imply a difference in stability on the two sides of the boundary. The dynamics can be unstable on both sides, or stable on both sides, but with very different dynamical properties on the two sides of the boundary. This can occur, even with the parameters being very close to the boundary on each side of the boundary.

In short, even with very high precision of parameter estimates, the nature of dynamics can be dramatically different within different subsets of the parameter estimates’ confidence region. Robustness of dynamical inferences is severely damaged, when a singular bifurcation boundary enters within the confidence region of a model’s parameter estimates.

2. **Bergstrom—Wymer Continuous Time UK Model**

2.1. **Background**

Among the models that have direct relevance to this research include high dimension continuous time macroeconometric models in Bergstrom, Nowman and Wymer (1992),

2.2. Introduction

In Part 2, we discuss several papers by Barnett and He on bifurcation analysis using Bergstrom, Nowman and Wymer’s continuous-time dynamic macroeconometric model of the UK economy. Barnett and He chose this policy-relevant model as their first to try, because the model is particularly well suited to these experiments. The model contains adjustment speeds producing Keynesian rigidities and hence possible Pareto improving policy intervention. In addition, as a system of second order differential equations, the model can produce interesting dynamics and possesses enough equations and parameters to permit it to be fitted plausibly to the UK data.

Barnett and He (1999) discovered that both saddle-node bifurcations and Hopf bifurcations coexist within the model’s region of plausible parameter setting. Bifurcation boundaries are located and drawn. The model’s Hopf bifurcation helps to provide explanations for some cyclical phenomena in the UK macroeconomy. The Barnett and He paper designed a numerical algorithm for locating the model’s bifurcation boundaries. That algorithm was provide above in section 1.3.4.

Barnett and He (1999) observed that stability of the model had not previously been tested. They found that the point estimates of the models parameters are outside the stable subset of the parameter space, but close enough to the bifurcation boundary so that the hypothesis of stability cannot be rejected. Confidence regions around the parameter estimates are intersected by the boundary separating stability from instability, with the point estimates being on the unstable side.
Barnett and He (2002) explored the problem of selection of a “stabilization policy.” The purpose of the policy was to bifurcate the system from an unstable to a stable operating regime by moving the parameters’ point estimates into the stable region. The relevant parameter space is the augmented parameter space, including both the private sector’s parameters and the parameters of the policy rule. They found that policies that would produce successful bifurcation to stability are difficult to determine, and the policies recommended by the originators of the model, based on reasonable economic intuition and full knowledge of their own model, tend to be counterproductive, since such policies contract the size of the stable subset of the parameter space and move that set farther away from the private sector’s parameter estimates. These results point towards the difficulty of designing successful countercyclical stabilization policy in the real world, where the structure of the economy is not accurately known. Barnett and He (1999) also proposed a new formula for determining the bifurcation boundary for transcritical bifurcations.

a. The Model

The model is described by the following 14 second-order differential equations.

\[ D^2 \log C = \gamma_1 (\lambda_1 + \lambda_2 - D \log C) + \gamma_2 \log \left[ \frac{\beta_1 e^{-\gamma_1(\beta_2 (r - D \log p) + \beta_3 D \log p)q(Q + P)}}{T_1 C} \right], \quad (2.1) \]

\[ D^2 \log L = \gamma_3 (\lambda_2 - D \log L) + \gamma_4 \log \left[ \frac{\beta_4 e^{-\gamma_1(\beta_6 K - \beta_6 - \beta_6 K)}}{L} \right], \quad (2.2) \]

\[ D^2 \log K = \gamma_3 (\lambda_1 + \lambda_2 - D \log K) + \gamma_6 \log \left[ \frac{\beta_5 (\frac{Q}{K})^{1+\beta_6}}{r - \beta_7 D \log p \beta_6} \right], \quad (2.3) \]

\[ D^2 \log Q = \gamma_7 (\lambda_1 + \lambda_2 - D \log Q) + \gamma_8 \log \left[ \frac{(1 - \beta_9 \left( \frac{q}{p} \right)^{1+\beta_6}}{Q} \right], \quad (2.4) \]

\[ D^2 \log p = \gamma_9 \left( D \log \left( \frac{w}{p} \right) \right) + \gamma_{10} \log \left[ \frac{\beta_1 \beta_4 T_2 e^{-\gamma_1(\beta_6 K - \beta_6)}}{p} \right], \quad (2.5) \]

\[ D^2 \log w = \gamma_{11} \left( \lambda_1 - D \log \left( \frac{w}{p} \right) \right) + \gamma_{12} D \log \left( \frac{p_i}{q} \right) + \gamma_{13} \log \left[ \frac{\beta_4 e^{-\gamma_1(\beta_6 K - \beta_6)}}{\beta_6 e^{\gamma_2 t}} \right], \quad (2.6) \]

\[ D^2 r = -\gamma_{14} Dr + \gamma_{15} \left[ \beta_{13} + r_f - \beta_{14} D \log q + \beta_{15} \frac{p(Q+P)}{M} - r \right], \quad (2.7) \]
\[ D^2 \log I = \gamma_16 \left( \lambda_1 + \lambda_2 - D \log \left( \frac{p(I)}{q_p} \right) \right) \]
\[ + \gamma_{17} \log \left[ \frac{\beta_9 \left( \frac{q_p}{p(I)} \right) \beta_{10} (C + G_c + DK + E_n + E_o)}{(p(I))} \right], \quad (2.8) \]

\[ D^2 \log E_n = \gamma_{18} (\lambda_1 + \lambda_2 - D \log E_n) + \gamma_{19} \log \left[ \frac{\beta_{16} \gamma_{17} \left( \frac{p(I)}{q_p} \right)^{\beta_{18}}}{E_n} \right], \quad (2.9) \]

\[ D^2 F = -\gamma_{20} DF + \gamma_{21} \left[ \beta_{19} (Q + P) - F \right], \quad (2.10) \]

\[ D^2 P = -\gamma_{22} DP + \gamma_{23} \left[ \beta_{20} + \beta_{21} (r_f - D \log p_f) \right] K_a - P \], \quad (2.11) \]

\[ D^2 K_a = -\gamma_{24} DK_a + \gamma_{25} \left[ \beta_{22} + \beta_{23} (r_f - r) - \beta_{24} D \log q - \beta_{25} d_x \right] (Q + P) - K_a \], \quad \text{(2.12)} \]

\[ D^2 \log M = \gamma_{26} (\lambda_3 - D \log M) + \gamma_{27} \log \left[ \frac{\beta_{26} \epsilon_{3x}}{M} \right] \]
\[ + \gamma_{28} D \log \left[ \frac{E_n + E_o + \rho - F}{(p(I))} \right] + \gamma_{29} \log \left[ \frac{E_n + E_o + \rho - F - DK_a}{(p(I))} \right], \quad (2.13) \]

\[ D^2 \log q = \gamma_{30} D \log (p_f / q_p) + \gamma_{31} \log \left[ \frac{\beta_{27} \epsilon_{3x}}{q_p} \right] + \gamma_{32} D \log \left[ \frac{E_n + E_o + \rho - F}{(p(I))} \right] \]
\[ + \gamma_{33} \log \left[ \frac{E_n + E_o + \rho - F - DK_a}{(p(I))} \right], \quad (2.14) \]

where \( t \) is time, \( D \) is the derivative operator, \( D x = \frac{dx}{dt}, D^2 x = \frac{d^2 x}{dt^2} \) and \( C, E_n, F, I, K, K_a, L, M, P, Q, q, r, w \) are endogenous variables whose definitions are listed below.

\( C \) = real private consumption,
\( E_n \) = real non-oil exports,
\( F \) = real current transfers abroad,
\( I \) = volume of imports,
\( K \) = amount of fixed capital,
\( K_a \) = cumulative net real investment abroad (excluding changes in official reserve),
\( L \) = employment,
\[ M \text{ = money supply,} \]
\[ P \text{ = real profits, interest and dividends from abroad,} \]
\[ p \text{ = price level,} \]
\[ Q \text{ = real net output,} \]
\[ q \text{ = exchange rate (price of sterling in foreign currency),} \]
\[ r \text{ = interest rate,} \]
\[ w \text{ = wage rate.} \]

The variables \( d_x, E_o, G_c, p_f, p_i, r_f, T_1, T_2, Y_f \) are exogenous variables with the following definitions:

\[ d_x \text{ = dummy variables for exchange controls (} d_x = 1 \text{ for 1974-79, } d_x = 0 \text{ for 1980 onwards),} \]
\[ E_o \text{ = real oil exports,} \]
\[ G_c \text{ = real government consumption,} \]
\[ p_f \text{ = price level in leading foreign industrial countries,} \]
\[ p_i \text{ = price of imports (in foreign currency),} \]
\[ r_f \text{ = foreign interest rate,} \]
\[ T_1 \text{ = total taxation policy variable defined by Bergstrom, Nowman, and Wymer (1992, p. 317),} \]
\[ T_2 \text{ = indirect taxation policy variable defined by Bergstrom, Nowman, and Wymer (1992, p. 317),} \]
\[ Y_f \text{ = real income of leading foreign industrial countries.} \]

The structural parameters \( \beta_i, i = 1,2, ..., 27, \gamma_j, j = 1,2, ..., 33, \) and \( \lambda_k, k = 1,2,3 \) can be estimated from historical data. A set of their estimates using quarterly data from 1974 to 1984 are given in Table 2 of Bergstrom, Nowman, and Wymer (1992). These equations are derived from economic theory. The exact interpretations of these 14 equations are available in Bergstrom, Nowman, and Wymer (1992).
Both endogenous and exogenous variables are time-varying quantities. The exogenous variables are assumed to satisfy the following conditions in equilibrium:

\[
\begin{align*}
    d_x &= 0, \\
    E_o &= 0, \\
    G_e &= g^*(Q + P), \\
    p_f &= p_f^*e^{\lambda_4 t}, \\
    p_i &= p_i^*e^{\lambda_4 t}, \\
    r_f &= r_f^*, \\
    T_1 &= T_1^*, \\
    T_2 &= T_2^*, \\
    Y_f &= Y_f^*e^{\left(\frac{\lambda_1+\lambda_2}{\beta_{17}}\right)t},
\end{align*}
\]

where \( g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^* \) and \( \lambda_4 \) are constants. According to Bergstrom, Nowman, and Wymer (1992), the assumptions are reasonable. Under such assumptions, it has been proven that \( C(t), ..., q(t) \) in (2.1)-(2.14) change at constant rates in equilibrium. Note that the system described by (2.1)-(2.14) is not autonomous, since time itself enters as an exogenous variable. To study the dynamics of the system around the equilibrium, Barnett and He (2002) make a transformation by defining a set of new variables \( y_1(t), y_2(t), ..., y_{14}(t) \):

\[
\begin{align*}
    y_1(t) &= \log\{C(t)/C^*e^{(\lambda_1+\lambda_2) t}\}, \\
    y_2(t) &= \log\{L(t)/L^*e^{\lambda_2 t}\}, \\
    y_3(t) &= \log\{K(t)/K^*e^{(\lambda_1+\lambda_2) t}\}, \\
    y_4(t) &= \log\{Q(t)/Q^*e^{(\lambda_1+\lambda_2) t}\}, \\
    y_5(t) &= \log\{p(t)/p^*e^{(\lambda_3-\lambda_1-\lambda_2) t}\}, \\
    y_6(t) &= \log\{w(t)/w^*e^{(\lambda_3-\lambda_2) t}\}, \\
    y_7(t) &= r(t) - r^*, \\
    y_8(t) &= \log\{I(t)/I^*e^{(\lambda_1+\lambda_2) t}\}, \\
    y_9(t) &= \log\{E_n(t)/E_n^*e^{(\lambda_1+\lambda_2) t}\}, \\
\end{align*}
\]
\[ y_{10}(t) = \log\{F(t)/F^*e^{(\lambda_1+\lambda_2)t}\}, \]
\[ y_{11}(t) = \log\{P(t)/P^*e^{(\lambda_1+\lambda_2)t}\}, \]
\[ y_{12}(t) = \log\{K_a(t)/K_{a^*}e^{(\lambda_1+\lambda_2)t}\}, \]
\[ y_{13}(t) = \log\{M(t)/M^*e^{\lambda_3t}\}, \]
\[ y_{14}(t) = \log\{q(t)/q^*e^{(\lambda_1+\lambda_2+\lambda_4-\lambda_3)t}\}, \]

where \( C^* \), \( L^* \), \( K^* \), \( Q^* \), \( p^* \), \( w^* \), \( r^* \), \( l^* \), \( E_n^* \), \( F^* \), \( P^* \), \( K_a^* \), \( M^* \), \( q^* \) are functions of the vector \((\beta, \gamma, \lambda)\) of 63 parameters in equations (2.1)-(2.14) and the additional parameters \( g^*, p_f^*, p_i^*, r_f^*, T_1^*, T_2^*, Y_f^*, \lambda_4 \).

The following is a set of differential equations derived from (2.1)-(2.14):

\[
D^2y_1 = -y_1Dy_1 + y_2\{\log(Q^*e^{y_4} + P^*e^{y_{11}}) - \log(Q^* + P^*) - \beta_2y_7 + (\beta_2 - \beta_3)Dy_5 - y_1\} \tag{2.15}
\]
\[
D^2y_2 = -y_3Dy_2 + y_4\left\{\frac{1}{\beta_6}\log\left[\frac{(Q^*)^{-\beta_6} - \beta_6(K^*)^{-\beta_6}}{(Q^*)^{-\beta_6} - \beta_6y_4 - \beta_6(K^*)^{-\beta_6}}\right] - y_2\right\} \tag{2.16}
\]
\[
D^2y_3 = -y_5Dy_3 + y_6\{(1 + \beta_6)(y_4 - y_3) + \log[r^* - \beta_7(\lambda_3 - \lambda_1 - \lambda_2) + \beta_9] - \log[y_7 + r^* - \beta_7(Dy_5 + \lambda_3 - \lambda_1 - \lambda_2) - \beta_8]\} \tag{2.17}
\]
\[
D^2y_4 = -y_7Dy_4 + y_8\{\log\left[\frac{1 - \beta_9\left(q^*p^*_f\right)\beta_1}{1 - \beta_9\left(q^*p^*_f\right)\beta_1}e^{\beta_1y_5 + y_4}\right]
+ \log(C^*e^{y_1} + g^*(Q^*e^{y_4} + P^*e^{y_{11}}) + K^*e^{y_3}(Dy_3 + \lambda_1 + \lambda_2) + E_n^*e^{y_9}) - \log(C^* + g^*(Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*) - y_4\}, \tag{2.18}
\]
\[
D^2y_5 = y_9(Dy_6 - Dy_5) + y_{10}\{y_6 - y_5 - \frac{1 + \beta_6}{\beta_6}\log\left[1 - \beta_5\left(q^*K^*_f\right)\beta_6e^{\beta_6(y_4 - y_3)}\right]
+ \frac{1 + \beta_6}{\beta_6}\log\left[1 - \beta_5\left(q^*K^*_f\right)\beta_6\right]\}, \tag{2.19}
\]
\[
D^2y_6 = y_{11}(Dy_5 - Dy_6) - y_{12}(Dy_5 + Dy_14) + y_{13}\left\{\frac{1}{\beta_6}\log[(Q^*)^{-\beta_6} - \beta_5(K^*)^{-\beta_6}]
- \frac{1}{\beta_6}\log[(Q^*)^{-\beta_6} - \beta_5(K^*)^{-\beta_6}e^{\beta_6y_3}]\right\}, \tag{2.20}
\]
\[ D^2 y_7 = -\gamma_{14} D y_7 + \gamma_{15} \left[ \beta_{15} \frac{p^* e^{y_5} (Q^* e^{y_4} + P^* e^{y_11})}{M^* e^{y_{13}}} - \beta_{15} \frac{p^* (Q^* + P^*)}{M^*} - \beta_{14} D y_{14} - y_7 \right], \quad (2.21) \]

\[ D^2 y_8 = \gamma_{16} (D y_5 + D y_{14} - D y_8) + \gamma_{17} \{ (1 + \beta_{16}) (y_5 + y_{14}) - y_8 \}
+ \log [C e^{y_5} + g^* (Q^* e^{y_4} + P^* e^{y_11}) + K^* e^{y_3} (D y_3 + \lambda_1 + \lambda_2)] + \frac{E^n e^{y_0}}{n e^{y_{11}}}, (2.22) \]

\[ D^2 y_9 = -\gamma_{18} D y_9 - \gamma_{19} \left[ \beta_{18} \frac{(y_5 + y_{14})}{F^* e^{y_{11}}} \right] \]

\[ D^2 y_{10} = -\{y_{20} + 2(\lambda_1 + \lambda_2)\} D y_{10} - (D y_{10})^2 + \gamma_{21} \beta_{19} \left[ \frac{Q^* e^{y_4} + P^* e^{y_11}}{F^* e^{y_{10}}} - \frac{Q^* + P^*}{F^*} \right], \quad (2.24) \]

\[ D^2 y_{11} = -\{y_{22} + 2(\lambda_1 + \lambda_2)\} D y_{11} - (D y_{11})^2 + \gamma_{23} \left[ \beta_{20} + \beta_{21} (r_f - \lambda_4) \right] \left[ \frac{K^*_a e^{y_{12}}}{P^* e^{y_{11}}} - K^*_a \right] \]

\[ D^2 y_{12} = -\{y_{24} + 2(\lambda_1 + \lambda_2)\} D y_{12} - (D y_{12})^2 + \gamma_{25} \left[ \beta_{22} + \beta_{23} (r_f - r^* - y_7) \right. \]

\[ D^2 y_{13} = -\gamma_{26} D y_{13} - \gamma_{27} y_{13} + \gamma_{28} \left[ \frac{E^n e^{y_9} D y_9 + P^* e^{y_{11}} D y_{11} - F^* e^{y_{10}} D y_{10}}{E^n e^{y_9} + P^* e^{y_{11}} - F_e^{y_{10}}} \right] \]

\[ D^2 y_{14} = -\gamma_{30} (D y_5 + D y_{14} - y_8) + \gamma_{31} (y_5 + y_{14}) \]

The equilibrium of the original system (2.1)-(2.14) corresponds to the equilibrium \( y_i = 0, i = 1, 2, ..., 14 \) of (2.15)-(2.18). The major advantage of the new system described by (2.15)-(2.18) is that it is autonomous, but still retains all the dynamic properties of the original system (2.1)-(2.14). Autonomous systems are the main subject of nonlinear systems theory. In general, non-autonomous system are difficult to analyze. In Barnett and He (1999), the paper analyzed
the local dynamics of (2.15)-(2.28) in a local neighborhood of the equilibrium \(y_i = 0, i = 1, 2, \ldots, 14\). For simplicity, the system (2.15)-(2.28) is denoted as

\[
\textbf{D}x = \textbf{f}(\textbf{x}, \theta),
\]

where

\[
x = [y_1 \ D y_1 \ y_2 \ D y_2 \ \ldots \ y_{14} \ D y_{14}]' \in R^{28}
\]
is the state vector, while

\[
\theta = [\beta_1, \ldots, \beta_{27}, \gamma_1, \ldots, \gamma_{33}, \lambda_1, \lambda_2, \lambda_3]' \in R^{63}
\]
is the parameter vector, and \(\textbf{f}(\textbf{x}, \theta)\) is a vector of functions of \(\textbf{x}\) and \(\theta\) obtained from (2.15)-(2.28). Every component of \(\textbf{f}(\textbf{x}, \theta)\) is smooth (infinitely differentiable) in a neighborhood of the origin. Note that (2.29) is a first-order system. The point \(x^* = 0\) is an equilibrium of (2.29). Since \(\theta\) represents physical quantities, its entries are bounded by theoretical and \textit{a priori} feasibility constraints [see, Table 2 of Bergstrom, Nowman, and Wymer (1992)]. Let \(\Theta\) denote the feasible region determined by those bounds. \(\Theta\) is a bounded region.

\[\textbf{b. Stability of the Equilibrium}\]

In section 1.2, the discussion on stability described a means to analyze local stability of the system through linearization. The linearized system of (2.15)-(2.28) is

\[
D^2y_1 = -y_1 D y_1 + y_2 \left\{Q^* e^{y_4 + p^* e^{y_{11}}} - \frac{Q^* e^{y_4 + p^* e^{y_{11}}}}{Q^* + p^*} - y_2 \right\},
\]

\[
D^2y_2 = -y_3 D y_2 + y_4 \left\{(Q^*)^{-\beta_6} y_4 - \frac{(Q^*)^{-\beta_6} y_4}{(Q^*)^{-\beta_6} - \beta_6} \right\} - y_2,
\]

\[
D^2y_3 = -y_5 D y_3 + y_6 \left\{(1 + \beta_6)(y_4 - y_3) - \frac{y_5 - \beta_7 D y_5}{r - \beta_7(\lambda_3 - \lambda_1 - \lambda_2)}\right\},
\]

\[
D^2y_4 = -y_7 D y_4 + y_8 \left\{-y_4 - \frac{\beta_9 \left(\frac{Q^* e^{y_4 + p^* e^{y_{11}}}}{Q^* + p^*}\right)^{\beta_{10}}}{1 - \beta_9 \left(\frac{Q^* e^{y_4 + p^* e^{y_{11}}}}{Q^* + p^*}\right)^{\beta_{10}}} \right\} + \frac{C^* y_1 + g^*(Q^* y_4 + p^* y_{11}) + K^* D y_3 + K^*(\lambda_1 + \lambda_2) y_3 + E_n y_9}{C^* + g^*(Q^* + p^* + K^*(\lambda_1 + \lambda_2) + E_n)},
\]

\[
D^2y_5 = y_9 (D y_5 - D y_5) + y_{10} \left\{(1 + \beta_6) \left(\frac{\beta_5 \left(\frac{Q^* e^{y_4 + p^* e^{y_{11}}}}{Q^* + p^*} \right)^{\beta_6}}{1 - \beta_5 \left(\frac{Q^* e^{y_4 + p^* e^{y_{11}}}}{Q^* + p^*} \right)^{\beta_6}} \right) \left(\frac{Q^* e^{y_4 + p^* e^{y_{11}}}}{Q^* + p^*} \right)^{\beta_6} \right\} + y_6 - y_5,
\]

\[
D^2y_6 = y_{11} (D y_5 - D y_5) - y_{12} (D y_5 + D y_{14}) + y_{13} \left\{(Q^*)^{-\beta_6} y_4 - \frac{(Q^*)^{-\beta_6} y_4}{(Q^*)^{-\beta_6} - \beta_6} \right\},
\]
\[ D^2 y_7 = -\gamma_{14} D y_7 + \gamma_{15} \{ -\beta_{14} D y_{14} - y_7 \}
+ \frac{\beta_{15}}{M^*} \left[ (Q^* + P^*)p^* (y_5 - y_{13}) + p^* (Q^* y_4 + P^* y_{11}) \right] \],
\quad (2.36)

\[ D^2 y_8 = \gamma_{16} (D y_5 + D y_{14} - D y_8) + \gamma_{17} \{ (1 + \beta_{10}) (y_5 + y_{14}) - y_8 \}
+ \frac{C^* y_1 + g^* (Q^* y_4 + P^* y_{11}) + K^*(\lambda_1 + \lambda_2) y_3 + K^* D y_3 + E_n^* y_9}{C^* + g^* (Q^* + P^*) + K^*(\lambda_1 + \lambda_2) + E_n^*}, \]
\quad (2.37)

\[ D^2 y_9 = -\gamma_{18} D y_9 - \gamma_{19} \{ \beta_{18} (y_5 + y_{14}) + y_9 \}, \quad (2.38) \]

\[ D^2 y_{10} = -\{ \gamma_{20} + 2(\lambda_1 + \lambda_2) \} D y_{10} + \frac{\gamma_{21} \beta_{19}}{F^*} \{ Q^* (y_4 - y_{10}) + P^* (y_{11} - y_{10}) \}, \quad (2.39) \]

\[ D^2 y_{11} = -\{ \gamma_{22} + 2(\lambda_1 + \lambda_2) \} D y_{11} + \gamma_{23} \{ \beta_{20} + \beta_{21} (r_f^* - \lambda_4) \} \frac{K_A^*}{P^*} (y_{12} - y_{11}), \quad (2.40) \]

\[ D^2 y_{12} = -\{ \gamma_{24} + 2(\lambda_1 + \lambda_2) \} D y_{12} + \gamma_{25} \{ -\beta_{24} \frac{Q^* + P^*}{K_A^*} D y_{14} - \beta_{23} \frac{Q^* + P^*}{K_A^*} y_7 \}
+ [\beta_{22} + \beta_{23} (r_f^* - r^*) - \beta_{24} (\lambda_1 + \lambda_2 + 4 - \lambda_3)] \frac{Q^* (y_{12} - y_{11}) + P^* (y_{11} - y_{12})}{K_A^*}, \quad (2.41) \]

\[ D^2 y_{13} = -\gamma_{26} D y_{13} - \gamma_{27} y_{13} + \gamma_{28} \left\{ \frac{E_n^* D y_5 + P^* D y_{11} - F^* D y_{10}}{E_n^* + P^* - F^*} + D y_5 + D y_{14} - D y_8 \right\}
+ \gamma_{29} \left\{ \frac{E_n^* y_5 + P^* y_{11} - F^* y_{10} - K_A^* (\lambda_1 + \lambda_2) y_{12} - K_A^* D y_{12}}{E_n^* + P^* - F^* - K_A^* (\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8 \right\}, \quad (2.42) \]

\[ D^2 y_{14} = -\gamma_{30} (D y_5 + D y_{14}) - \gamma_{31} (y_5 + y_{14})
+ \gamma_{32} \left\{ \frac{E_n^* D y_9 + P^* D y_{11} - F^* D y_{10}}{E_n^* + P^* - F^*} + D y_5 + D y_{14} - D y_8 \right\}
+ \gamma_{33} \left\{ \frac{E_n^* y_5 + P^* y_{11} - F^* y_{10} - K_A^* (\lambda_1 + \lambda_2) y_{12} - K_A^* D y_{12}}{E_n^* + P^* - F^* - K_A^* (\lambda_1 + \lambda_2)} + y_5 + y_{14} - y_8 \right\}, \quad (2.43) \]

In matrix form, these equations become

\[ \dot{x} = A(\theta)x. \quad (2.44) \]

For the set of estimated values of \{\beta_i\}, \{\gamma_j\}, and \{\lambda_k\} given in Table 2 of Bergstrom, Nowman, and Wymer (1992), all the eigenvalues of \(A(\theta)\) are stable (having negative real parts) except the following three:

\[ s_1 = 0.0033, \quad s_2 = 0.009 + 0.0453i, \quad s_3 = 0.009 - 0.0453i. \]

Since the real parts of the unstable eigenvalues are so small and near zero, it is unclear whether they are caused by errors in estimation or the structural properties of the system itself.
Next, Barnett and He (2002) examine the statistical significance of the inference of instability. For each $\theta_i, i = 1, 2, ..., 63$, its estimate and the corresponding standard error are provided in Table 2 of Bergstrom, Nowman, and Wymer (1992). For a given confidence level $p$, a confidence interval can be obtained for any $\theta_i$:

$$[\hat{\theta}_i - e_p \sigma_i, \hat{\theta}_i + e_p \sigma_i]$$

where $\hat{\theta}_i$ and $\sigma_i$ are respectively the estimate and standard error of parameter $\theta_i$, and $e_p$ is the standard normal percentile, as is consistent with the distributional assumptions in Bergstrom, Nowman, and Wymer (1992). Both $\hat{\theta}_i$ and $\sigma_i$ are available in Table 2 of Bergstrom, Nowman, and Wymer (1992). For example, the 95% confidence interval for $\theta_1 = \beta_1$ is [0.9324, 0.9476]. Let $[\bar{\theta}_i, \bar{\theta}_i]$ denote the confidence interval for parameter $\theta_i, i = 1, 2, ..., 63$. For several parameters, the estimates were on the boundaries of the theoretical feasible intervals. In this case, Barnett and He (2002) assume $\sigma_i = 0$, so confidence interval becomes one point, the estimate. There are 8 such parameters: $\theta_8, \theta_{11}, \theta_{13}, \theta_{28}, \theta_{33}, \theta_{48}, \theta_{60}, \theta_{63}$. Hence our inferences condition upon their corner solution values. Define

$$\Theta_1 = \{ \theta \in \Theta | \theta_i \in [\bar{\theta}_i, \bar{\theta}_i], i = 1, 2, ..., 63 \}. $$

Then $\Theta_1 \subset \Theta$ denotes the region of $\theta$ determined by the Cartesian product of those confidence intervals. Any change in the stability of (2.29) over $\Theta_1$ implies that Barnett and He (2002) cannot reject the hypothesis that the bifurcation boundary might cross the confidence region.

Barnett and He (1999) found a parameter vector $\theta^* \in \Theta$ such that (2.44) is stable. From this $\theta^*$ they could find a stable region of $\theta$ and the boundaries of bifurcations. They used the gradient method to find a $\theta^*$ such that all eigenvalues of $A(\theta^*)$ have strictly negative real parts. To find such a $\theta^*$, they consider the following problem of minimizing the maximum real parts of eigenvalues of matrix $A(\theta)$:

$$\min_{\theta \in \Theta} R_{\text{max}}(A(\theta)) \tag{2.45}$$

where

$$R_{\text{max}}(A(\theta)) = \max_i \{\text{real}(\lambda_i): \lambda_1, \lambda_2, ..., \lambda_{28} \text{ are eigenvalues of } A(\theta)\}.$$ 

Since the dimension of $A(\theta)$ is 28, which is relatively high, Barnett and He (1999) could not acquire a closed-form expression for $R_{\text{max}}(A(\theta))$. They used the gradient method to solve the minimization problem (2.45). More precisely, let $\theta^{(n)}$ be the estimated set of parameter values given in Table 2 of Bergstrom, Nowman, and Wymer (1992). At step $n, n \geq 0$, with $\theta^{(n)}$, let
\[ \theta^{(n+1)} = \theta^{(n)} - a_n \frac{\partial R_{\text{max}}(A(\theta))}{\partial \theta} \bigg|_{\theta = \theta^{(n)}}, \]

where \( \{a_n, n = 0,1,2, \ldots \} \) is a sequence of (positive) step sizes. After several iterations (20 iterations in this case), the algorithm converged to the following point, \( \theta^* \in \Theta_1 \),

\[
\theta^* = [0.9400, 0.2256, 2.3894, 0.2030, 0.2603, 0.1936, 0.1829, 0.0183, 0.2470, -0.2997, 1.0000, 23.5000, -0.0100, 0.1260, 0.0082, 13.5460, 0.4562, 1.0002, 0.0097, 0.0049, 0.2812, -0.1000, 44.9030, 0.1260, 0.0082, 13.5460, 0.4562, 1.0002, 0.0097, 0.0049, 0.2812, -0.1000, 44.9030, 0.1260, 0.0082, 13.5460, 0.4562, 1.0002, 0.0097, 0.0049, 0.2812, -0.1000, 44.9030, 0.1260, 0.0082, 13.5460, 0.4562, 1.0002, 0.0097, 0.0049, 0.2812, -0.1000, 44.9030, 0.1260, 0.0082, 13.5460, 0.4562, 1.0002].
\]

The corresponding \( R_{\text{max}}(A(\theta^*)) = -0.0039 \), implying that all eigenvalues of \( A(\theta^*) \) have strictly negative real parts and the system (2.44) is locally asymptotically stable around \( x^* = 0 \) for \( \theta^* \). This suggests that Barnett and He (1999) could not reject the hypothesis of stability.

One interesting fact is that if the confidence level is reduced to 90%, which results in smaller confidence intervals, the algorithm failed to find a value of \( \theta \) under which the system (2.29) is stable. This seems to suggest that, with 90% confidence level, the system (2.29) is unstable for all parameters \( \theta \in \Theta_1 \), and Barnett and He (1999) could not accept the hypothesis of stability.

c. Determination of Bifurcation Boundaries

The goal of this section is to find bifurcation boundaries of the model. Since the linearized system (2.44) only deals with local stability of the system, the existing papers on the model deal with local bifurcations as opposed to global bifurcations.

In the previous section we have seen that \( A(\theta) \) has three eigenvalues with strictly positive real parts for the set of parameters given in Table 2 of Bergstrom, Nowman, and Wymer (1992). However Barnett and He (1999) also found when \( \theta = \theta^* \), all eigenvalues of (2.44) have strictly negative real parts. Since eigenvalues are continuous functions of entries of \( A(\theta) \), there must exist at least one eigenvalue of \( A(\theta) \) with zero real part on the bifurcation boundary. Different types of bifurcations may occur according to the way unstable eigenvalues
are created. A new method of finding transcritical bifurcations is also proposed in Barnett and He (2002).

i. Saddle-node and Hopf Bifurcations

A saddle-node bifurcation occurs when a system has a nonhyperbolic equilibrium with a geometrically simple zero eigenvalue at the bifurcation point and additional transversality conditions are satisfied (given by the Sotomayor’s (1973) Theorem).

When \( \det(A(\theta)) = 0 \), \( A(\theta) \) has at least one zero eigenvalue. If \( A(\theta) \) has exactly one simple zero eigenvalue, under additional technical transversality conditions, this point corresponds to a saddle-node bifurcation. So the first condition used to find the bifurcation boundary is

\[
\det(A(\theta)) = 0.
\]

Note that \( A(\theta) \) is a sparse matrix. Analytical forms of bifurcation boundaries can be obtained for most parameters. To demonstrate the feasibility of this approach, Barnett and He (1999) consider finding the bifurcation boundaries for \( \beta_2 \) and \( \beta_5 \).

Theorem 2.1  The bifurcation boundary for \( \beta_2 \) and \( \beta_5 \) is determined by

\[
1.36\beta_2\beta_5 + 21.78\beta_5 - 2.05\beta_2 - 10.05 = 0.
\]

Proof. (Barnett and He (1999)) Denote \( A(\theta) = [a_{i,j}] \).

Barnett and He (1999) observed from (2.30)-(2.43) that only the following entries of \( A(\theta) \) are functions of \( \beta_2 \) and \( \beta_5 \). All other entries do not depend on \( \beta_2 \) and \( \beta_5 \).

\[
a_{2,10} = -\gamma_2 (\beta_2 - \beta_3), \quad a_{2,13} = -\gamma_2 \beta_2,
\]

\[
a_{4,5} = -\gamma_4 \frac{(K^*)^{-\beta_6} \beta_5}{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6} \beta_5}, \quad a_{4,7} = \gamma_4 \frac{(Q^*)^{-\beta_6}}{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6} \beta_5},
\]

\[
a_{10,5} = -\gamma_{10} (1 + \beta_6) \frac{\beta_5 (Q^*)^{\beta_6}}{1 - \beta_5 (Q^*)^{\beta_6}}, \quad a_{10,7} = \gamma_{10} (1 + \beta_6) \frac{\beta_5 (Q^*)^{\beta_6}}{1 - \beta_5 (Q^*)^{\beta_6}},
\]

\[
a_{12,5} = -\gamma_{12} \frac{(K^*)^{-\beta_6} \beta_5}{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6} \beta_5}, \quad a_{12,7} = \gamma_{12} \frac{(Q^*)^{-\beta_6}}{(Q^*)^{-\beta_6} - (K^*)^{-\beta_6} \beta_5}.
\]
Setting parameter values at $\theta^*$ except for $\beta_2$ and $\beta_5$, Barnett and He (1999) obtained from direct calculation that

$$\det(A) = -4.63 - 1 \times 10^{-15} - 2 \times 10^{-16} \beta_2 + 2.178 \times 10^{-15} \beta_5 + 1.36 \times 10^{-16} \beta_2 \beta_5.$$

Hence, (2.46) follows from setting $\det(A(\theta)) = 0$. □

For Hopf Bifurcation, Barnett and He (1999) use the methods in section 1.3.4., e.g. Procedure (P1), to find Hopf bifurcation.

ii. Transcritical Bifurcations

Recall the definition and transversality condition for transcritical bifurcations from section 1.3.1, if $A(\theta)$ has exactly one simple zero eigenvalue under the transversality condition (equation (1.4), section 1.3.1), this $\theta$ corresponds to a transcritical bifurcation. So the condition to find such a bifurcation boundary is

$$\det(A(\theta)) = 0.$$

Analytical forms of bifurcation boundaries can be obtained for most parameters. For example, if we are interested in bifurcations when two parameters $\theta_i, \theta_j$ change, while others are kept at $\theta^*$, the matrix $A(\theta)$ may be rewritten as

$$A(\theta) = A(\theta^*) + B(\theta^*)D(\mu)C(\theta^*), \quad (2.47)$$

where $\mu = [\theta_i, \theta_j]$, and $D(\mu)$ is a matrix of appropriate dimension. The dimension of $D(\mu)$ is usually much smaller than that of $A(\theta)$. In this case, the following proposition is helpful for simplifying the determination of transcritical bifurcation boundaries.

**Proposition 2.1.** Assume that $A(\theta)$ has structure (2.47) and that all eigenvalues of $A(\theta^*)$ have strictly negative real parts. Then $\det(A(\theta)) = 0$ if and only if

$$\det \left( I + D(\mu)C(\theta^*)A^{-1}(\theta^*)B(\theta^*) \right) = 0. \quad (2.48)$$

Proof. See Barnett and He (2002), Proposition 1, P13.

Proposition 2.1 is useful for simplifying the calculation of $\det(A(\theta))$. To demonstrate the usefulness of this approach, consider finding the bifurcation boundary for $\mu = [\theta_2, \theta_{23}] = [\beta_2, \beta_{23}]$. Only the following entries of $A(\theta)$ are functions of $\mu$.

$$a_{2,10}(\mu) = \gamma_2(\beta_2 - \beta_3), \quad a_{2,13}(\mu) = -\gamma_2 \beta_2,$$
\[ a_{24,7}(\mu) = \frac{\gamma_{25} \delta Q^*}{K_2}, \quad a_{24,13}(\mu) = -\frac{\gamma_{25} \beta_{23} (Q^* + P^*)}{K_2}, \]
\[ a_{24,21}(\mu) = \frac{\gamma_{25} \delta P^*}{K_2}, \quad a_{24,23}(\mu) = -\frac{\gamma_{25} \delta (Q^* + P^*)}{K_2}, \]

where \( \delta = \beta_{22} + \beta_{23} (r_f - r^*) - \beta_{24} (\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3) \). In this case, \( B(\theta^*) \in \mathbb{R}^{28 \times 2} \) has all zero entries except that its (2,1) entry is 1 and its (24,2) entry is 1. The matrix \( C(\theta^*) \in \mathbb{R}^{5 \times 28} \) also has zero entries, except the entries are 1 at the following locations: (1,7), (2,10), (3,13), (4,21), (5,23). The matrix \( D(\mu) \) is

\[
D(\mu) = d(\mu) - d(\theta^*),
\]

where

\[
d(\mu) = \begin{bmatrix}
0 & a_{2,10}(\mu) & a_{2,13}(\mu) & 0 & 0 \\
a_{24,7}(\mu) & 0 & a_{24,13}(\mu) & a_{24,21}(\mu) & a_{24,23}(\mu)
\end{bmatrix}.
\]

Direct calculation yields

\[
C(\theta^*) A^{-1}(\theta^*) B(\theta^*) = \begin{bmatrix}
13.7090 & -17.1187 \\
0 & 0 \\
-1.7276 & 2.1573 \\
-616.4935 & 389.2039 \\
-616.4935 & 389.2039
\end{bmatrix}.
\]

Using Proposition 2.1, Barnett and He (1999) observed that \( det(A) = 0 \) is equivalent to

\[
det \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + D(\mu) \right) \begin{bmatrix}
13.7090 & -17.1187 \\
0 & 0 \\
-1.7276 & 2.1573 \\
-616.4935 & 389.2039 \\
-616.4935 & 389.2039
\end{bmatrix} = 0,
\]

or equivalently,

\[
-14.23 + 15.91 \theta_2 + 0.28 \theta_{23} - 0.50 \theta_2 \theta_{23} = 0.
\]

Stability of the system (2.29) when parameters take values on the bifurcation boundary needs to be determined by examining the higher order terms in \( D x = A(\theta) x + F(x, \theta) \). This is usually done with the help of center manifold theory. After appropriate coordinate transformation, it is possible to write \( D x = A(\theta) x + F(x, \theta) \) as [see Glendinning (1994) or Guckenheimer and Holmes (1983)]:

\[
D x_1 = A_1(\theta) x_1 + F_1(x_1, x_2, \theta), \tag{2.49}
\]
\[ Dx_2 = A_2(\theta)x_2 + F_2(x_1, x_2, \theta), \]  

(2.50)

where all eigenvalues of \( A_1(\theta) \) have zero real parts and all eigenvalues of \( A_2(\theta) \) have strictly negative real parts. Center manifold theory says that there exists a center manifold \( x_2 = h(x_1) \) such that

\[ h(0) = 0 \text{ and } Dh(0) = 0. \]

Substituting \( x_2 = h(x_1) \) into (49), Barnett and He (1999) obtain

\[ Dx_1 = A_1(\theta)x_1 + F_1(x_1, h(x_1), \theta). \]  

(2.51)

The stability of (2.29) is connected to that of (2.51) through the following theorem.

**Theorem 2.2** [Henry (1981), Carr(1981)] If the origin of (2.51) is locally asymptotically stable (respectively unstable), then the origin of (2.29) is also locally asymptotically stable (respectively unstable).

Substituting \( x_2 = h(x_1) \) into (2.50), Barnett and He (1999) observed that \( h(x_1) \) satisfies

\[ Dx_2 = Dh(x_1)dx_1 = Dh(x_1)[A_1(\theta)x_1 + F_1(x_1, h(x_1), \theta)] \]

\[ = A_2(\theta)h(x_1) + F_2(x_1, h(x_1), \theta), \]

or \( h(x_1) \) satisfies

\[ Dh(x_1)[A_1(\theta)x_1 + F_1(x_1, h(x_1), \theta)] = A_2(\theta)h(x_1) + F_2(x_1, h(x_1), \theta), \]  

(2.52)

\[ h(0) = 0, Dh(0) = 0. \]  

(2.53)

The equations (2.52) and (2.53) can be used to solve or approximate, at least in principle, \( h(x_1) \). In practice, solving (2.52) and (2.53) would be difficult. One usually uses a Taylor series approximation of \( h(x_1) \) with several terms to determine the local asymptotic stability or instability of (2.51). For most cases, especially codimension one bifurcations, the dimension of (2.51) is usually one or two. In the case of transcritical bifurcations, the dimension of (2.51) is one. In this case, let

\[ F_1(x_1, x_2, \theta) = a_1 \frac{x_1^2}{2!} + a_2 x_2 + a_3 \frac{x_1^3}{3!} + \cdots, \]

\[ F_2(x_1, x_2, \theta) = b_1 \frac{x_1^2}{2!} + b_2 x_2 + b_3 \frac{x_1^3}{3!} + \cdots. \]

Assume that \( h(x_1) \) has the following Taylor expansion

\[ h(x_1) = \alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \cdots. \]
Then (2.52) becomes

\[
\left( ax_1 + \beta \frac{x_1^2}{2!} + \cdots \right) A_1(\theta)x_1 + a_1 \frac{x_1^2}{2!} + x_1a_2 \left( \alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \cdots \right) + a_3 \frac{x_1^3}{3!} + \cdots = A_2(\theta) \left( \alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \cdots \right) + b_1 \frac{x_1^2}{2!} + x_1b_2 \left( \alpha \frac{x_1^2}{2!} + \beta \frac{x_1^3}{3!} + \cdots \right) + b_3 \frac{x_1^3}{3!} + \cdots.
\]

By comparing coefficients of the same order terms and also observing that \( A_1(\theta) = 0 \) at a bifurcation point, Barnett and He (1999) observed that

\[
\alpha = -A_2^{-1}(\theta)b_1, \quad \beta = A_2^{-1}(\theta)(\alpha a_1 - b_2 a).
\]

Therefore, (2.51) becomes

\[
Dx_1 = A_1(\theta)x_1 + a_1 \frac{x_1^2}{2!} + \left( \frac{a_2}{2!} + \frac{a_3}{3!} \right) x_1^3 + \cdots \quad (2.54)
\]

The stability analysis of (2.54) determines the stability characteristics of \( Dx = A(\theta)x + F(x, \theta) \).

As an example, consider the stability of the system on the transcritical bifurcation boundary for parameters \( \beta_2, \beta_{23} \). The stability of the system (2.29) could be determined using the previously described approach. For example, consider the point \((\beta_2, \beta_{23}) = (0.1068, 55.9866)\) on the boundary. Barnett and He (1999) found that (2.51) becomes

\[
Dx_1 = 0.1308 x_1^2 + o(x_1^2),
\]

which is locally asymptotically unstable at \( x_1 = 0 \). Therefore, it follows from center manifold theory that the system (2.29) is locally asymptotically unstable at this transcritical bifurcation point.

d. Stabilization Policy

We have seen in the previous section that both transcritical and Hopf bifurcation exist in the UK continuous time macroeconometric model. In this section, we provide Barnett and He’s (2002) results investigating the control of bifurcations using fiscal feedback laws. They define stabilization policy to be intentional movement of bifurcation regions through policy intervention, with the intent of moving the stable region to include the parameters. If the parameters were inside the stable region without policy, then there would be no need for stabilization policy.

We first consider the effect of a heuristically plausible fiscal policy of the following form, as suggested in Bergstrom, Nowman, and Wymer (1992):
\[ D \log T_1 = \gamma \left[ \beta \log \left( \frac{Q}{Q^* \epsilon (\lambda_1 + \lambda_2)t} \right) - \log \left( \frac{T_1}{T_1^*} \right) \right]. \] (2.55)

The control feedback rule (2.55) adjusts the fiscal policy instrument, \( T_1 \), towards a partial equilibrium level, which is an increasing function of the ratio of output to its steady state level. In (2.55), \( \beta \) is a measure of the strength of the feedback, and \( \gamma \) governs the speed of adjustment. By choosing appropriate parameters \( \beta, \gamma \), it was found in Bergstrom, Nowman, and Wymer (1992) that the control law (2.55) can reduce the positive real parts of unstable eigenvalues, implying that the policy might be stabilizing. Since Bergstrom is perhaps the foremost authority on the UK continuous time model and on such continuous time macroeconometric models in general, we would expect that if any heuristically plausible fiscal policy would be successful, it would be Bergstrom’s. However, Barnett and He (2002) found that the control law (2.55) is unlikely to stabilize the systems (2.1)-(2.14).

Define \[ y_{15} = \log \left( \frac{T_1}{T_1^*} \right). \]

Then it can be verified that \( y_{15} \) satisfies \[ D y_{15} = \gamma \beta y_4 - \gamma y_{15}. \]

Adding this equation to the system (2.29), Barnett and He (2002) obtain \[ Dw = A'(\theta)w + F'(x, \theta), \] (2.56)

where \[ w = \begin{bmatrix} x \\ y_{15} \end{bmatrix}, \quad F'(x, \theta) = \begin{bmatrix} F(x, \theta) \\ 0 \end{bmatrix} \]

and \( A'(\theta) \) is the corresponding coefficient matrix.

Consider three sets of parameter values of \( \gamma : \beta = 0.04, \gamma = 0.02; \beta = 0.01, \gamma = 0.05; \beta = 0, \gamma = 0 \). The case \( \beta = 0, \gamma = 0 \) corresponds to the original system (2.1)-(2.14), in which no fiscal policy control is applied. The graph showing the effect of a simple fiscal policy of three cases indicates that some stable regions could be destabilized and some unstable regions could be stabilized. Since the feasible region is smaller under control than without control, the policy is not likely to succeed.

Consider a more sophisticated fiscal control policy, based upon optimum control theory. Let the control be
\[ u = \log \left( \frac{T_1}{T_1^*} \right), \quad (2.57) \]

Under the control (2.46), the system (2.29) becomes
\[ D_x = A(\theta)x + Bu + F(x, \theta), \quad (2.58) \]
where \( B = [0 \ - \gamma_2 \ 0 \ ... \ 0]^T \in \mathbb{R}^{28} \). Direct verification yields that the controllability matrix \([B \ A B \ A^{27} B]\) has rank 7, implying that the pair \((A, B)\) is not controllable. Therefore, it is not possible to set the closed-loop eigenvalues of the coefficient matrix of (2.58) arbitrarily. However, a numerical analysis shows that there exists a linear transformation \( z = T x \) such that
\[
D_z = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix} z + \begin{bmatrix}
0 \\
B_2
\end{bmatrix} u,
\]
where \( A_{11} \in \mathbb{R}^{21 \times 21}, A_{21} \in \mathbb{R}^{7 \times 21}, A_{22} \in \mathbb{R}^{7 \times 7}, B_2 = [0 \ ... \ 0 \ 1] \in \mathbb{R}^7, \)
\[ TA(\theta)T^{-1} = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}, \quad TB = \begin{bmatrix}
0 \\
B_2
\end{bmatrix}, \]
and \((A_{22}, B_2)\) is controllable. The exact numerical procedure for this decomposition can be found, for example, in Khalil (1992). Further, all eigenvalues of \( A_{11} \) have negative real parts, implying that \((A(\theta), B)\) is stabilizable.

To obtain a feedback control law stabilizing (2.58), Barnett and He (2002) consider solving the problem of minimizing
\[ J = \int_0^\infty [x^T U x + V u^2] dt, \]
where \( U \in \mathbb{R}^{28 \times 28} \) and \( V \in \mathbb{R}^1 \) are positive definite. It is known from linear system theory that the optimal feedback control law is given by
\[ u = K x, \quad K = -V^{-1}B^T P, \]
where \( P \) is positive definite and solves the algebraic Riccati equation
\[ PA + A^T P - PBV^{-1}B^T P + U = 0. \]
Choosing \( U = I \) and \( V = 1 \), Barnett and He (2002) got
\[ K = [1.5036, 0.4754, 0.0178, 0.0307, -1.1897, 18.5851, 7.2979, 1.9063, 2.3147, 23.2392, 0.7488, 7.2091, 38.9965, 39.4000, 0.1841, 0.2129, 0.3061, 0.0494, -0.0027, 0.0000, -0.0013, -0.0002, 0.9550, 1.8482, -0.3329, -0.5475, 0.9369, -1.0402]. \quad (2.59) \]
Under the control \( u = Kx \), the equation (2.58) becomes

\[
\dot{x} = (A(\theta) + BK)x + F(x, \theta).
\]  

(2.60)

The choice of \( K \) ensures that all the eigenvalues of \( A + BK \) have strictly negative real parts. Therefore, the state feedback law \( u = Kx \) indeed stabilizes the system (2.60). Direct verification confirms that there exist no bifurcations under the control law (2.60) for \((\beta_2, \beta_5)\).

Barnett and He (2002) further check the stability of (2.60) under the control law (2.60) for all parameter \( \theta \in \Theta \). The purpose is to see if there is a parameter \( \theta' \in \Theta \) at which the system (2.60) is unstable. The following \( \theta' \in \Theta_1 \) has been found

\[
\theta' = [0.9400, 0.5074, 2.0913, 0.2030, 0.2612, 0.1933, 0.2309, 0.0000, 0.2510, -0.3423, 1.0000, 23.5000, -0.0100, 0.2086, 0.0332, 13.5460, 0.4562, 0.9322, 0.0100, 0.0034, 0.1324, -0.5006, 100.0000, 0.0000, 0.0004, 71.4241, 0.8213, 4.0000, 1.0289, 0.3631, 0.1201, 0.1000, 0.0010, 3.7015, 0.4860, 1.1270, 0.0042, 3.3994, 0.4802, 0.1300, 0.6851, 0.0620, 1.2134, 0.3830, 4.0000, 3.2535, 3.8592, 4.0000, 4.0000, 3.5723, 0.4775, 0.0071, 0.6104, 0.0143, 0.1718, 0.1227, 2.5551, 0.1833, 0.0035, 0.0000, 0.0018, 0.0004, 0.0100].

The corresponding \( R_{\text{max}}(A(\theta')) = 0.4971 \). Therefore, there indeed exists a parameter \( \theta' \in \Theta_1 \) at which (2.60) is unstable.

Because of the Lucas critique, the problems associated with using structural models for policy simulations are well known. In addition, the possibility of time inconsistency of optimal control policy conditionally upon a structural model is well known. While the use of Euler equation models having deep parameters is to be preferred for policy simulations. Nevertheless, it is interesting to ask whether the use of control feedback policy with a structural model would be easily implemented, if the Lucas critique and time inconsistency issues did not exist. It seems often to be assumed that such active policy easily could be designed, if it were not for the problems produced by the Lucas critique and by the time inconsistency of optimal control.

However, the results here indicate that even without those problems, the design of a successful feedback policy can be difficult. Even when the structural parameters of the other equations remain constant, adjoining a policy feedback rule to a system causes bifurcation boundaries to shift. The policy is successful, if those shifts cause the stable region to move towards the actual values of the parameters sufficiently to include the parameters within the stable region. Barnett and He (2002) found that with the UK continuous time model, the
selection of a fiscal policy feedback rule from Bergstrom’s particularly “well educated” heuristic economic reasoning is counterproductive. While the use of optimal control theory is successful, conditionally upon the model, the resulting policy equation is too complicated to be of practical use and is heavily dependent upon the model. Furthermore, the negative results from the heuristic non-optimal policy raise serious questions about the robustness of the optimal control conclusion to specification error. In addition, the success of the optimal control policy abstracts from the problems of possible time inconsistency of optimal control policy.

In short, the effects of policy feedback rules depend upon the complicated geometry of bifurcation boundaries and how they are moved by augmentation of the model by the feedback rule. It is not at all unlikely that such policies, when applied in the real world, could prove to be counterproductive, even if the Lucas critique and time inconsistency were not problems.

3. Leeper and Sims Model

3.1. Background

Grandmont (1985) found that the parameter space of even the simplest, classical general-equilibrium macroeconomics models are stratified into bifurcation regions. This result changed the prior common view that different kinds of economic dynamics can only be produced by different kinds of structures. But his result is based on the model in which all policies are Ricardian equivalent, no frictions exist, employment is always full, competition is perfect, and all solutions are Pareto optimal. Hence he was not able to reach conclusions about the policy relevance of his dramatic discovery. However, the econometric implications of Grandmont’s findings are particularly important, if bifurcation boundaries cross the confidence regions surrounding parameter estimates in policy-relevant models. Stratification of a confidence region into bifurcated subsets seriously damages robustness of dynamical inferences.

The dramatic transformation of views precipitated by Grandmont’s paper was criticized for lack of policy relevance. As a result, Barnett and He (1999, 2001a, 2001b, 2002) investigated a continuous-time traditional Keynesian structural model and found results supporting Grandmont’s conclusions, details of which are shown in section 2 above. Barnett and He found transcritical, codimension-two, and Hopf bifurcation boundaries within the parameter space of the Bergstrom-Wymer continuous-time dynamic macroeconometric model of the UK economy. The model contains frictions through adjustment lags, displays reasonable dynamics fitting the UK economy’s data, and is clearly policy relevant. See Bergstrom and Wymer (1976), Bergstrom (1996), Bergstrom, Nowman, and Wandasiewicz (1994), Bergstrom, Nowman, and Wymer (1992), and Bergstrom and Nowman (2006). Barnett and He found that bifurcation boundaries
cross confidence regions of parameter estimates in that model, such that both stability and instability are possible within the confidence regions.

The Lucas critique has motivated development of Euler-equations general equilibrium macroeconometric models. Hence, Barnett and one of his coauthors continued the investigation of policy relevant bifurcation by searching the parameter space of the best known of the policy relevant Euler-equations macroeconometric models: the path-breaking Leeper and Sims (1994) model. The results further confirm Grandmont’s views, but with the finding of an unexpected form of bifurcation: singularity bifurcation. Although known in engineering and mathematics, singularity bifurcation has not previously been encountered in economics. Barnett and He (2004, 2006) have made clear the mathematical nature of singularity bifurcation and why it is likely to be common in the class of modern Euler equation models rendered important by the Lucas critique.

In all of these studies, the models used are highly policy-relevant and were not modified from their influential previously-published form. While Grandmont’s model has been criticized for its lack of policy relevance, we believe from the accumulating evidence that Grandmont’s conclusions are correct and are highly relevant to policy. In particular, these results cast into doubt the robustness of dynamical inferences acquired in the traditional manner by simulating macroconometric models solely at their parameter point-estimates. To be able to achieve robustness of dynamical inferences, such simulations should be made at various settings throughout the parameters’ confidence region.

3.2. Introduction

Leeper and Sims (1994) Euler equations stochastic-dynamic general-equilibrium model is intended to address such issues as the Lucas critique (Lucas (1976)) for the US economy. It has deep parameters that are invariant to policy rule changes and thereby immune to the Lucas critique. Similar models are developed in Kim (2000) and others, but the Leeper and Sims model was the seminal model in that literature.

The dimension of the state space in the Leeper and Sims model is substantially lower than in the Bergstrom, Norman, and Wymer UK model. However, the dimension is still too high for complete analysis by generally available analytical approaches. By numerical methods complementing theoretical analysis, Barnett and He (2006b, 2008) find that the dynamics of the Leeper and Sims model are complicated by the model’s structure as an Euler equations model, since such models usually have no closed form algebraic solution. It should be pointed out that the Leeper and Sims model indeed has several attractive features, such as the integration of several factors of interest including monetary shocks to price level, wage level, and interest
rates. The model also simultaneously considers behaviors of consumers, firms, and the government. Furthermore, the model treats monetary and fiscal policies explicitly.

Our analysis reveals the existence of a singularity bifurcation boundary within a small neighborhood of the estimated parameter values. Barnett and He (2006b, 2008) discover that the order of the dynamics of the Leeper and Sims model could change within a small neighborhood of the estimated parameter values. When the parameter values approach the singularity boundary, as parameters change within a small neighborhood of the estimated parameter values, one eigenvalue of the linearized part of the model can move quickly from finite to infinite and back again to finite. Such phenomenon characterizes the instability of the structure inherent in the model. A large stable eigenvalue characterizes the case in which some variables can respond rapidly to changes of other variables, while a large unstable eigenvalue corresponds to the case in which rapid movement occurs of one variable away from other variables. Infinity eigenvalue implies existence of pure algebraic relationships among the variables. This sensitivity to the setting of the parameters presents serious challenges to the robustness of dynamical inferences. The source of the problem is the nature of the mapping from the Euclidean parameter space to the function space of dynamical solutions.

On the singularity boundary, the number of differential equations will decrease, while the number of algebraic constraints will increase. Such change in the order of dynamics had not previously been found with macroeconometric models. But Barnett and He (2006b, 2008) find from the relevant theory that singularity bifurcation may be a common property of Euler equations models and illustrated that theoretical speculation with the well-known Leeper and Sims’ model from that class. The dramatic implications of singularity bifurcation are not limited to the change in the dimension of the dynamics on the bifurcation boundary. The nature of the dynamics on one side of a singularity bifurcation boundary is very different from the nature of the dynamics on the other side, although of the dimension of the dynamics is the same on both sides.

Although the Leeper and Sims model is specified as a closed economy model, it is implicitly open economy as estimated by Leeper and Sims, since the US data used in the model includes imported and exported goods. A logical extension of this experiment would be to apply the procedures to an explicitly open-economy Euler-equations model. Because of the connection between Euler-equation implicit functions and singularity bifurcation, we would expect similar results with an explicitly open-economy Euler-equations model. In section 6, we will survey research on bifurcation phenomena in open-economy New Keynesian models.

3.3. The Model
Leeper and Sims (1994) derive their macroeconometric model by considering consumers, firms, and the government. Both consumers and firms maximize their respective objective functions. The government provides monetary and tax policies to satisfy intertemporal government budget identify and the pursuit of countercyclical policy objectives. The detailed derivation of the models is available in Leeper and Sims (1994) and will not be repeated in this survey as Barnett and He (2006b, 2008) are interested in the structural properties of the model. In the next section, we introduce the mathematical equations of the model and investigate its structural properties.

The Leeper and Sims model consists of the following 12 state variables.

\[ L = \text{labor supply}, \]
\[ C^* = \text{consumption net of transaction costs}, \]
\[ M = \text{consumer demand for non-interest-bearing money}, \]
\[ D = \text{consumer demand for interest-bearing money}, \]
\[ K = \text{capital}, \]
\[ Y = \text{factor income from capital and labor, excluding interest on government debt}, \]
\[ C = \text{gross consumption}, \]
\[ Z = \text{investment}, \]
\[ X = \text{consumption goods aggregate price}, \]
\[ Q = \text{investment goods price}, \]
\[ V = \text{income velocity of money}, \]
\[ P = \text{general price level}. \]

The model assumes that the consumer maximizes

\[ E \left[ \int_0^\infty \exp(-\int_0^t \beta(s) ds) \frac{(C^* \pi (1-L)^{1-\pi})^{1-\gamma}}{1-\gamma} dt \right] \]

subject to

\[ XC + QZ + \tau + \frac{\dot{M} + \dot{D}}{P} = Y + \frac{iD}{P}, \]
\[ XC^* + \phi VY = XC, \]
\[ \dot{K} = Z - \delta K, \]
\[ Y = rK + wL + S, \]
\[ V = \frac{PY}{M}, \]

where \( \pi \in (0,1) \) and \( \gamma > 0 \) are parameters; \( 0 \leq \beta(s) \leq 1 \) is the subjective rate of time preference at time \( s \), \( \tau \) is the level of lump-sum taxes paid by the representative consumer; \( i \) is the nominal rate of return earned on government bonds; and \( S \) is the sum of dividends received by
the representative consumer, $w$ is the wage rate; $\varphi > 0$ is the transaction cost per unit of $VY$; $\delta \geq 0$ is the rate of depreciation of capital; and $r$ = rental rate of return on capital. Parameters in this stochastic dynamic general-equilibrium model are not necessarily assumed to be constant or deterministic.

The firms' optimization problem is

$$\max \left\{ X(C + g) + QI^* + A(\alpha K^\sigma + L^\sigma)^{\frac{1}{\sigma}} - rK - wL - ((C + g)\mu + \theta I^* \mu)^{\frac{1}{\mu}} \right\},$$

where $g$ is the level of government purchases. The following are parameters:

- $A > 0, \alpha > 0, \theta > 0, \mu \geq 0,$ and $0 \leq \sigma \leq 1.$

Investment goods produced by the firm, $I^*$, include both those bought by the existing population, $Z$, and those purchased by the government for distribution to the newborn. Thus, a market-clearing condition is $I^* = Z + nK$, where $n = \text{the fraction of existing capital purchased by the government for distribution to the newborn.}$

In this model, the state variables satisfy the following differential equations:

$$\frac{1}{P}(\dot{M} + \dot{D}) = Y - XC - QZ + \frac{iD}{P} + \tau,$$  \hspace{0.5cm} (3.1)

$$\dot{K} = Z - \delta K,$$  \hspace{0.5cm} (3.2)

$$\left(1 - \pi(1 - \gamma)\right)\frac{\dot{c}^*}{c^*} + \left(1 - \gamma\right)(1 - \pi)\frac{L}{1 - L} + \frac{\dot{x}}{x} + \frac{\dot{p}}{P} = i - \beta + \frac{\pi}{\pi} + \hat{\pi}(1 - \gamma)\log\left(\frac{c^*}{c}\right),$$  \hspace{0.5cm} (3.3)

$$\frac{\dot{p}}{p} + \frac{\dot{Q}}{Q} = i + \delta - (1 - 2\varphi V)\frac{r}{Q},$$  \hspace{0.5cm} (3.4)

where (3.1) represents the consumers’ budget constraint, (3.2) is the law of motion for capital, and ((3.3),(3.4)) are first-order conditions from optimizing consumers’ objective function. In addition to the four dynamic equations, the state variables also satisfy the following algebraic constraints.

$$X = \left(\frac{Y}{C + g}\right)^{1-\mu},$$  \hspace{0.5cm} (3.5)

$$Q = \theta\left(\frac{Y}{Z + nK}\right)^{1-\mu},$$  \hspace{0.5cm} (3.6)

$$r = A^\sigma \alpha \left(\frac{Y}{K}\right)^{1-\delta},$$  \hspace{0.5cm} (3.7)

$$w = A^\sigma \left(\frac{Y}{L}\right)^{1-\delta},$$  \hspace{0.5cm} (3.8)

$$XC^* + \varphi VY = XC,$$  \hspace{0.5cm} (3.9)

$$Y = rK + wL + S,$$  \hspace{0.5cm} (3.10)
\[ V = \frac{PY}{M}, \quad (3.11) \]
\[ X(C + g) + Q(Z + nK) = Y, \quad (3.12) \]
\[ (1 - 2\phi V) \frac{w}{\chi} = \frac{1 - \pi}{\pi} \frac{C^*}{1 - L}, \quad (3.13) \]
\[ i = \phi V^2. \quad (3.14) \]

The relations (3.5)-(3.8) are obtained from the first-order conditions by maximizing the firms’ objective function. Equation (3.9) defines consumption net of transaction costs, with total output serving as a measure of the level of transactions at a given point in time. Equation (3.10) defines income. Equation (3.11) is the income velocity of money. Equation (3.12) is the social resources constraint. Equations (3.13)-(3.14) are obtained from the first-order conditions for the consumers’ decision.

The control variables are the government policy variables, consisting of the nominal rate of return on government bonds, \( i \), and the level of lump-sum taxes, \( \tau \). Leeper and Sims (1994) introduced the following monetary and tax policies into the model. The monetary policy rule is
\[
\frac{1}{i} \frac{di}{dt} = a_p \log \left( \frac{p}{\bar{p}} \right) + a_{int} \frac{\dot{p}}{p} + a_i \log \left( \frac{i}{\bar{i}} \right) + a_L \log \left( \frac{L}{\bar{L}} \right) + \epsilon_i, \quad (3.15)
\]
and the tax policy is
\[
\frac{d}{dt} \frac{\tau}{C} = b_T \left( \frac{\tau}{C} - \frac{\bar{\tau}}{C} \right) + b_L \log \left( \frac{L}{\bar{L}} \right) + b_{int} \frac{\dot{p}}{p} + b_x \frac{D}{PY} - \frac{\bar{D}}{\bar{PY}} + \epsilon_\tau. \quad (3.16)
\]

The overscored variables denote steady state values, so that \( \bar{D}/\bar{Y} \) is the steady state debt-to-income level, where income is measured by Leeper and Sims as GNP. The free parameters are \( \bar{D}/\bar{Y} \), the steady state price level, \( \bar{P} \), the \( a \)'s, and the \( b \)'s. The disturbance noises are \( \epsilon_i \) and \( \epsilon_\tau \).

In this model, it is conventional to use \( \tau_c = \frac{\tau}{C} \), rather than \( \tau \), as a control. Therefore, the control variables are \( i \) and \( \tau_c \). The parameters and exogenous variables, \( n, g, \pi, \delta, \theta, \alpha, A \) and \( \phi \), are specified by Leeper and Sims to follow logarithmic first-order autoregressive (AR) processes in continuous time, while \( \beta \) is specified to be a logarithmic first-order AR in unlogged form. However, Barnett and He (2006b, 2008) analyze the structural properties of (3.1)-(3.14) without external disturbances. As a result, in equation (3.3), Barnett and He (2006b, 2008) set \( \pi = 0 \) and treat \( \pi \) as a fixed parameter, along with the model’s other parameters, which are all treated as fixed. They treat the exogenous variables as realized at their measured values. The extension of this analysis to the case of stochastic bifurcation is a subject for future research.
The original form (3.1)-(3.14) has 12 state variables and 14 equations. For analytical investigation, it is best to have as few state variables as possible. For this purpose, Barnett and He (2006b, 2008) reduce the dimension of the problem by temporarily eliminating some state variables. They contract to the following 7 state variables

\[
x = \begin{bmatrix}
D \\
P \\
C \\
L \\
K \\
Z \\
Y
\end{bmatrix}.
\]

(3.17)

The remaining state variables can be written as unique functions of \(x\). By eliminating \(M, C^*, V, Q, X\) from the independent state variables, it can determined directly from (3.1)-(3.14) that \(x\) satisfies the following equations.

\[
\frac{1}{p} \dot{D} + \frac{\sqrt{\phi}}{p} \dot{P} + \left(\frac{\phi}{\sqrt{v}} \right) \dot{Y} = Y + \frac{iD}{p} - \frac{\theta}{C+g} \phi \mu \left(\frac{Y}{Z+nK}\right)^{1-\mu} L - \tau c \phi \mu + \frac{\sqrt{\phi}}{2v^{2} \phi \mu}.
\]

(18)

\[
(1 - \pi(1 - \gamma)) \left( \frac{1 - \phi V Y^{\mu}}{C - \theta V Y^{\mu}(C + g)^{1-\mu}} - \frac{1 - \mu}{C + g} \right) \dot{C} - \left( \frac{1 - \pi(1 - \gamma) \phi V \mu Y^{\mu-1}(C + g)^{1-\mu}}{C - \phi V Y^{\mu}(C + g)^{1-\mu}} + \frac{1 - \mu}{Y} \right) \dot{Y} + \frac{\dot{P}}{p} + \frac{(1 - \gamma)(1 - \pi)}{1 - \gamma} L = i3. - \beta + \frac{\phi \mu \gamma^{\mu-1}}{C - \phi V Y^{\mu}(C + g)^{1-\mu}} \frac{1}{2 \sqrt{v} \phi \mu}.
\]

(3.19)

\[
\frac{\dot{P}}{p} + (1 - \mu) \left( \frac{\gamma}{Y} - \frac{Z+nK}{Z+nK} \right) = -(1 - 2 \phi V) \alpha \gamma^{\mu-\alpha} (Z + nK)^{1-\mu} K^{\sigma-1} + i + \delta,
\]

(3.20)

\[
\dot{K} = Z - \delta K,
\]

(3.21)

\[
0 = (C + g)^{\mu} + \theta (Z + nK)^{\mu} - Y^{\mu},
\]

(3.22)

\[
0 = a K^{\sigma} + \sigma L - \sigma^{-\alpha} Y^{-\sigma}
\]

(3.23)

\[
0 = (1 - 2 \phi V) \alpha \gamma^{\mu-\alpha} (C + g)^{1-\mu} L^{1-\sigma} + \frac{1-\pi}{\pi} \phi V Y^{\mu}(C + g)^{1-\mu} - \frac{1-\pi}{\pi} \frac{C}{1-L}.
\]

(3.24)

For the ease of notation, Barnett and He (2006b, 2008) denote equations (3.18)-(3.24) as

\[h(x, u) \dot{x} = f(x, u),\]

(3.25)
\[ 0 = g(x, u) \]  

where \( x \) is the state vector, \( u \) is the vector of controls, \( h(x, u) \) is a matrix having dimension 4×7, and \( f(x, u) \) is a 4×1 vector. The dimension of the vector of functions \( g(x, u) \) is 3×1. Equation (3.25) describes the nonlinear dynamical behavior of the model, and (3.26) represents the algebraic constraints, which are nonlinear. Many systems can be described in the form of (3.25) and (3.26). Models in that form are called nonlinear descriptor systems in the mathematical literature on nonlinear dynamics.

Barnett and He (2006b,2008) use \( m, m_1, m_2, \) and \( l \) (with \( m = m_1 + m_2 \)), to denote respectively the dimension of \( x \), the number of differential equations in (3.25), the number of algebraic constraints in (3.26), and the dimension of the vector of control variables, \( u \). With the Leeper and Sims model, \( m = 7, m_1 = 4, m_2 = 3 \), and \( l = 2 \).

The steady state of the system (3.25)-(3.26) for the 7 state variables, \( x \), conditionally on the setting of the controls, \( u \), can be solved from the following equations:

\[ 0 = f(x, u), \]  

\[ 0 = g(x, u). \]  

The steady state of \( x \) and \( u \) are denoted by \( \bar{x} \) and \( \bar{u} \), respectively, where \( \bar{u} \) is found from (3.15) and (3.16) in the steady state to be

\[ \bar{\ell} = \beta, \]
\[ \bar{\tau} = 0, \]
\[ \bar{\tau}_c = \frac{\bar{\tau}}{C}. \]  

In particular, the first equation of (3.29) is found from (3.15) in the steady state, the second equation from the definition of steady state, and the third equation from (3.16) in the steady state. The values \( \bar{x} \) and \( \bar{u} \) are solutions to (3.27)-(3.28), and (3.29). The resulting steady state is the equilibrium of (3.25)-(3.26), when the control variables are set at their steady state.

The vector of parameters in the steady state system is

\[ p = [\pi \ \beta \ \theta \ \alpha \ a \ \phi \ \delta \ \mu \ \gamma \ \sigma]' \]

where the prime denotes transpose. Leeper and Sims (1994) estimate the parameters with quarterly data from 1959 to 1992. Although \( g \) is not a parameter of tastes or technology, it is
taken as a fixed value by the private sector at its setting by the government. The constraints on the parameter values and $g$ are:

$$0 < \pi < 1, \gamma > 0, 0 \leq \sigma \leq 1, \mu \geq 1, \delta \geq 0, 0 \leq \beta \leq 1, \delta > 0, g \geq 0.$$  \hspace{1cm} (3.30)

### 3.4. Singularity in Leeper and Sims Model

The structural properties of the Leeper and Sims model in a small neighborhood of the equilibrium $(\bar{x}, \bar{u})$ can be studied using local linearization around this equilibrium. The linearized system of (3.25) and (3.26) is

$$E_1 \dot{x} = A_1 x + B_1,$$  \hspace{1cm} (3.31)

$$0 = A_2 x + B_2 u,$$  \hspace{1cm} (3.32)

where

$E_1 = h(\bar{x}, \bar{u}) \in R^{m_1 \times m} = R^{4 \times 7},$

$A_1 = \left. \frac{\partial f(x,u)}{\partial x} \right|_{x=\bar{x}, u=\bar{u}} \in R^{m_1 \times m} = R^{4 \times 7},$

$A_2 = \left. \frac{\partial g(x,u)}{\partial x} \right|_{x=\bar{x}, u=\bar{u}} \in R^{m_2 \times m} = R^{3 \times 7},$

$B_1 = \left. \frac{\partial f(x,u)}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} \in R^{m_1 \times l} = R^{4 \times 2},$

$B_2 = \left. \frac{\partial g(x,u)}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} \in R^{m_2 \times l} = R^{3 \times 2}.$

The linearized system (3.31)-(3.32) is solvable, if it is regular. Using the relevant regularity condition from Gantmacher (1974), the following is the solvability condition, which must hold from some values of the determinant’s parameter, $s$:

$$det\left( \begin{bmatrix} sE_1 - A_1 \\ -A_2 \end{bmatrix} \right) \neq 0.$$  

If the regularity condition is violated for all $s$, the linearized system either has multiple solutions or no solution. Barnett and He (2006b, 2008) randomly chose parameter values within theoretically feasible region and observed that the Leeper and Sims model, as expected, is regular.

To study the structural properties of the Leeper and Sims model, Barnett and He (2006b, 2008) further transform the linearized system (3.31)-(3.32) into the following form.

**Definition 3.1** Two systems
\[ \mathbf{E}\dot{x} = \mathbf{A}x + \mathbf{B}u \]  
(3.33)

and

\[ \mathbf{E}\dot{y} = \mathbf{A}y + \mathbf{B}u \]  
(3.34)

are said to be restricted system equivalent (r.s.e), if there exist two nonsingular matrices \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \) such that

\[ \mathbf{T}_1\mathbf{E}\mathbf{T}_2 = \mathbf{E}, \quad \mathbf{T}_1\mathbf{A}\mathbf{T}_2 = \mathbf{A}, \quad \mathbf{T}_1\mathbf{B} = \mathbf{B}, \quad x = \mathbf{T}_2y. \]

The form (3.34) can be obtained by using the coordinate transform \( x = \mathbf{T}_2y \) into (3.33) and then multiplying both sides of (3.33) by \( \mathbf{T}_1 \) from the left. The relationship of r.s.e. allows one to transform a system into a convenient form, while preserving important properties of the system.

Barnett and He (2006b,2008) next transformed (3.31)-(3.32) into suitable r.s.e. forms. Denote

\[ \mathbf{D}\mathbf{E} = \mathbf{D}_{\mathbf{a}_{\mathbf{n}_{\mathbf{k}}}}(\mathbf{E}_1). \]

where \( \mathbf{D}\mathbf{E} \in \{1,2,3,4\} \). Then there exist nonsingular matrices \( \mathbf{T}_1 \in \mathbb{R}^{4\times4} \) and \( \mathbf{T}_2 \in \mathbb{R}^{7\times7} \) such that

\[ \mathbf{T}_1\mathbf{E}_1\mathbf{T}_2 = \begin{bmatrix} \mathbf{I}_{r_E} & 0 \\ 0 & 0 \end{bmatrix}. \]

which is a \( 4 \times 7 \) matrix.

Consider the following coordinate transform:

\[ x = \mathbf{T}_2 \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \]

where \( \mathbf{y}_1 \in \mathbb{R}^{r_E} \) and \( \mathbf{y}_2 \in \mathbb{R}^{m-r_E} = \mathbb{R}^{7-r_E} \). Substituting the form of \( x \) into (3.31)-(3.32) and also multiplying both sides of (3.31) by \( \mathbf{T}_1 \), it follows that (3.31)-(3.32) is r.s.e to

\[ \dot{\mathbf{y}}_1 = \mathbf{A}_{11}\mathbf{y}_1 + \mathbf{A}_{12}\mathbf{y}_2 + \mathbf{B}_{11}\mathbf{u}, \]  
(3.35a)

\[ 0 = \mathbf{A}_{21}\mathbf{y}_1 + \mathbf{A}_{22}\mathbf{y}_2 + \mathbf{B}_{12}\mathbf{u}, \]  
(3.35b)

\[ 0 = \mathbf{A}_{31}\mathbf{y}_1 + \mathbf{A}_{32}\mathbf{y}_2 + \mathbf{B}_{2}\mathbf{u}, \]  
(3.35c)

where
\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = T_1 A_1 T_2, \quad \begin{bmatrix}
B_{11} \\
B_{12}
\end{bmatrix} = T_1 B_1, \quad \begin{bmatrix}
A_{31} & A_{32}
\end{bmatrix} = A_2 T_2.
\]

Here note that \(A_{11} \in \mathbb{R}^{r_x \times r_E}, A_{12} \in \mathbb{R}^{r_E \times (7-r_E)}, A_{21} \in \mathbb{R}^{(4-r_E) \times r_E}, A_{22} \in \mathbb{R}^{(4-r_E) \times (7-r_E)}, A_{31} \in \mathbb{R}^{3 \times r_E}, A_{32} \in \mathbb{R}^{3 \times (7-r_E)}, B_{11} \in \mathbb{R}^{r_E \times 2}, \) and \(B_{12} \in \mathbb{R}^{(4-r_E) \times 2}, \) while \(y_1\) is an \(r_E\) dimensional vector and \(y_2\) is a \(7-r_E\) dimensional vector.

Combining equations (3.35a) and (3.35b), the following is acquired:

\[\dot{y}_1 = A_{11} y_1 + A_{12} y_2 + B_{11} u,\]  \hspace{1cm} (3.36a)
\[0 = \tilde{A}_{21} y_1 + \tilde{A}_{22} y_2 + \tilde{B}_{12} u,\]  \hspace{1cm} (3.36b)

where

\[
\tilde{A}_{21} = \begin{bmatrix} A_{21} \\
A_{31} \end{bmatrix}, \quad \tilde{A}_{22} = \begin{bmatrix} A_{22} \\
A_{32} \end{bmatrix}, \quad \tilde{B}_{12} = \begin{bmatrix} B_{12} \\
B_2 \end{bmatrix}.
\]

If \(\tilde{A}_{22}\) is nonsingular, it is possible to solve for \(y_2\) from the algebraic constraint equation (3.36b). In fact, in this case, it follows that

\[y_2 = - (\tilde{A}_{22})^{-1} (\tilde{A}_{21} y_1 + \tilde{B}_{12} u).\]

Substituting the form of \(y_2\) into (3.36a), it follows that

\[\dot{y}_1 = (A_{11} - A_{12} \tilde{A}_{22}^{-1} \tilde{A}_{21}) y_1 + (B_{11} - A_{12} \tilde{A}_{22}^{-1} \tilde{B}_{12}) u,
\]

or equivalently,

\[\dot{y}_1 = C y_1 + D u,\]  \hspace{1cm} (3.37)

where \(C = A_{11} - A_{12} \tilde{A}_{22}^{-1} \tilde{A}_{21} \in \mathbb{R}^{r_E \times r_E}\) and \(D = B_{11} - A_{12} \tilde{A}_{22}^{-1} \tilde{B}_{12} \in \mathbb{R}^{r_E \times 2}.\) Hence, if \(\tilde{A}_{22}\) is nonsingular, the dynamics of \(y_1\) can be explained entirely in terms of the system of ordinary differential equations, (3.37). The algebraic relationship between \(y_1\) and \(y_2\) in equation (3.36b) is needed solely to determine the derived dynamics of \(y_2.\)

However, this transformation would not be possible if \(\tilde{A}_{22}\) were singular. Only when \(\tilde{A}_{22}\) is nonsingular does it follow that the untransformed linear system ((3.31),(3.32)) is equivalent to ((3.37),(3.36c)). Setting of the parameters of \(\tilde{A}_{22}\) that cause the matrix to become singular produces a “singularity bifurcation” boundary within the parameter space. As explained in Barnett and He (2002,2004,2006), the dimension of dynamics change, when parameters move onto the bifurcation boundary. But if instead the parameters cross that boundary, the nature of the dynamics will change dramatically without changing the dimension of the dynamics. Consequently, the dynamics of the system ((3.31),(3.32)) could be
dramatically different from those of ordinary linear differential equations, if \( \mathbf{A}_{22} \) were singular. The dynamics also would change substantially, if \( \mathbf{A}_{22} \) moves between two settings located on opposite sides of a singular bifurcation boundary. The dimension of the dynamics will change only if \( \mathbf{A}_{22} \) becomes exactly singular, putting the model directly onto a singularity bifurcation boundary.

To see what could happen when \( \mathbf{A}_{22} \) is singular, the linearized system (3.36a) and (3.36b) could be rewritten as

\[
\begin{bmatrix}
\mathbf{I}_r & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{y}_1 \\
\mathbf{y}_2
\end{bmatrix}
= \begin{bmatrix}
\mathbf{A}_{11} & \mathbf{A}_{12} \\
\mathbf{A}_{21} & \mathbf{A}_{22}
\end{bmatrix}
\begin{bmatrix}
\mathbf{y}_1 \\
\mathbf{y}_2
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{B}_{11} \\
\mathbf{B}_{12}
\end{bmatrix}
\mathbf{u}.
\]  

(3.38)

If the Leeper and Sims model is regular, so is the matrix pair

\[
\begin{bmatrix}
\mathbf{I}_r & 0 \\
0 & 0
\end{bmatrix},
\begin{bmatrix}
\mathbf{A}_{11} & \mathbf{A}_{12} \\
\mathbf{A}_{21} & \mathbf{A}_{22}
\end{bmatrix}
\]

which is called a matrix pencil. For a regular matrix pencil, there exist nonsingular matrices \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \) such that (Gantmacher (1974))

\[
\mathbf{T}_1 \begin{bmatrix}
\mathbf{I}_r & 0 \\
0 & 0
\end{bmatrix}
\mathbf{T}_2
= \begin{bmatrix}
\mathbf{I}_{m_1} & 0 \\
0 & \mathbf{N}
\end{bmatrix}
\text{ and } \mathbf{T}_1
\begin{bmatrix}
\mathbf{A}_{11} & \mathbf{A}_{12} \\
\mathbf{A}_{21} & \mathbf{A}_{22}
\end{bmatrix}
\mathbf{T}_2
= \begin{bmatrix}
\mathbf{\tilde{A}}_1 & 0 \\
0 & \mathbf{I}_{m_2}
\end{bmatrix},
\]

where \( m_1 + m_2 = m \) and \( \mathbf{N} \) is a nilpotent matrix, i.e. there exists a positive integer \( d \geq 1 \) such that \( \mathbf{N}^d = 0 \). The smallest such integer \( d \) is called the nilpotent index of \( \mathbf{N} \). Clearly \( \mathbf{N} = 0 \) satisfies the definition of nilpotence. The following is another example of a nilpotent matrix:

\[
\mathbf{N}
= \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}.
\]  

(3.39)

Consider the coordinate transform

\[
\begin{bmatrix}
\mathbf{y}_1 \\
\mathbf{y}_2
\end{bmatrix}
= \mathbf{T}_2
\begin{bmatrix}
\mathbf{z}_1 \\
\mathbf{z}_2
\end{bmatrix}.
\]

Substituting for \( \mathbf{y} \) in equation (3.38) and multiplying both sides of (3.38) by \( \mathbf{T}_1 \) from the left, another r.s.e. form of ((3.31),(3.32)) appears:

\[
\dot{\mathbf{z}}_1 = \mathbf{\tilde{A}}_1 \mathbf{z}_1 + \mathbf{\tilde{B}}_1 \mathbf{u},
\]

(3.40)

\[
\mathbf{N} \dot{\mathbf{z}}_2 = \mathbf{z}_2 + \mathbf{\tilde{B}}_2 \mathbf{u},
\]

(3.41)
where

\[
\begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix} = \bar{T}_1 \begin{bmatrix}
B_{11} \\
B_{12}
\end{bmatrix}.
\]

The solutions to (3.40) and (3.41) are respectively

\[
z_1 = e^{\bar{A}_1(t-t_0)}z_1(0) + \int_{t_0}^{t} e^{\bar{A}_1(t-\xi)} \bar{B}_1 u(\xi) d\xi,
\]

\[
z_2 = -\sum_{k=1}^{d-1} \delta^{(k-1)}(t)N^k z_2(0) - \sum_{k=0}^{d-1} N^k \bar{B}_2 u^{(k)}(t),
\]

where \(t_0 \geq 0\) is the initial time, \(\delta^{(k-1)}(t)\) is the derivative of order \(k - 1\) of the Dirac delta function, and \(u^{(k)}\) denotes that \(k\)-th order derivative of \(u\).

Unless \(N = 0\) or the initial state \(z_2(0) = 0\), there exist impulsive terms in the first summation in the solution for \(y_2\), as well as the smooth derivative terms of \(u\) in the second summation. In fact when \(N = 0\), the above solution for \(z_2\) does not apply, although the solution for \(z_1\) above remains valid. This solution structure with nonzero \(N\) is very different from that of ordinary differential equations, such as (3.40), for \(z_1\).

The first summation in the solution for \(z_2\) could produce shock effects to the state response of \(z_2\). The Dirac delta, which is \(\delta^{(k-1)}(t)\) when \(k = 1\), is often called the unit impulse function. But if \(N = 0\), it follows from (3.41) that

\[
z_2 = -\bar{B}_2 u,
\]

which is a smooth algebraic relationship between \(z_2\) and \(u\). This bifurcation phenomenon at \(N = 0\) is consistent with the following theorem, providing equivalence between bifurcation at \(N \neq 0\) and at singularity of \(\bar{A}_{22}\).

**Theorem 3.1** If both (3.40)-(3.41) and (3.36a)-(3.36b) are r.s.e forms of the same linearized system (3.31)-(3.32), then \(N = 0\), if and only if \(\bar{A}_{22}\) is nonsingular, i.e.

\[
det(\bar{A}_{22}) \neq 0.
\]

**Proof.** See Barnett and He (2008), Theorem 3.1.

**Theorem 3.2** Assume that \(E_1\) has full row rank, i.e.

\[
\text{rank } (E_1) = m_1.
\]

Then \(\bar{A}_{22}\) is nonsingular, if and only if \(m \times m\) matrix \(\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}\) is nonsingular, i.e.
Proof. See Barnett and He (2008), Theorem 3.2.

Therefore, the following condition for singularity bifurcation is provided by Theorem 3.2:

\[ \det \left( \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \right) = 0. \]

Note that \( x_2 \) is solvable from (3.37) alone, if \( \bar{A}_{22} \) is nonsingular. Therefore, the singularity condition implies the case in which \( x_2 \) is not readily solvable from the algebraic (3.37) alone. The dynamic constraint (3.36) needs to be taken into account.

The following property on singularity condition will be relevant.

Corollary 3.1. Consider the following system describing the dynamics of \((x, v)\), where \( v \in \mathbb{R}^{m_3} \) for arbitrary \( m_3 \).

\[
\begin{align*}
E_1 \dot{x} + E_1 v &= A_1 x + A_1 v + B_1 u, \\
\dot{v} &= A_v v + B_v u, \\
0 &= A_2 x + A_2 v + B_2 u,
\end{align*}
\]

(3.42a) (3.42b) (3.42c)

where \( E_1, A_1, A_v, B_v, A_2 \) are arbitrary matrices of dimension 
\( m_1 \times m_3, m_1 \times m_3, m_3 \times m_3, m_3 \times l, \) and \( m_2 \times m_3, \)
respectively, and the other matrices are as defined above. Then the singularity condition for (3.42a), (3.42b), and (3.42c) is the same as that for (3.31)&(3.32).

Proof. See Barnett and He (2008), Corollary 3.1.

Corollary 3.1 says that adding (or deleting) state variables that can be modeled by ordinary differential equations does not change the singularity condition. This property is useful in reducing the dimension of the problem under consideration. For example, the Leeper and Sims’ model’s state variable, \( K \), could be dropped from the state vector, (3.17), in the system ((3.31),(3.32)), without affecting the singularity condition.

It is easy to verify that, after dropping the state variable \( K \), the singularity condition becomes
\[
\det\left( \begin{bmatrix} E'_1 \\ A'_2 \end{bmatrix} \right) = 0, \tag{3.43}
\]

in which

\[
E'_1 = \begin{bmatrix}
\frac{1}{p} & Y \\
0 & \frac{1}{p} \\
0 & \frac{1}{p}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{p} & PV \\
0 & (1 - \gamma)(1 - \pi) \\
0 & 1 - L
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
- \frac{1 - \mu}{Z + nK}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - \mu \\
1 - \mu
\end{bmatrix}
\]

and

\[
A'_2 = \begin{bmatrix}
0 & 0 & \mu(C + g)\mu^{-1} & 0 & \theta \mu(Z + nK)\mu^{-1} & \mu Y\mu^{-1} \\
0 & 0 & a_{23} & a_{24} & 0 & a_{26} \\
0 & 0 & 0 & \sigma L^{\sigma - 1} & 0 & A^{-\sigma} Y^{\sigma - 1}
\end{bmatrix}
\]

with

\[
e_{23} = \frac{1 - \pi(1 - \gamma)}{C^*} [1 - \phi YY\mu(C + g)\mu^{-2}] - \frac{1 - \mu}{C + g},
\]

\[
e_{26} = \frac{1 - \pi(1 - \gamma)}{C} [-\phi YY\mu(C + g)]\mu^{-1} + \frac{1 - \mu}{Y},
\]

\[
a_{23} = (1 - 2\phi V) A^\sigma Y^{\mu - \sigma} L^{\sigma - 1}(1 - \mu)(C + g)^{-\mu} - \frac{1 - \pi}{\pi} \frac{1}{1 - L},
\]

\[
a_{24} = (1 - 2\phi V) A^\sigma Y^{\mu - \sigma}(\sigma - 1)L^{\sigma - 2}(C + g)^{1 - \mu} - \frac{1 - \pi}{\pi} \frac{C}{(1 - L)^2},
\]

\[
a_{26} = (1 - 2\phi V) A^\sigma (\mu - \sigma) Y^{\mu - \sigma - 1} L^{\sigma - 1}(C + g)^{1 - \mu}.
\]

Note the prime does not designate transpose but rather deletion of the state variable, \( K \), from the vector \( x \) in equation (3.17) and deletion of equation (3.21), which is the corresponding differential equation for capital.

Direct calculation shows that (3.43) is equivalent to

\[
\det\left( \begin{bmatrix}
e_{23} & (1 - \gamma)(1 - \pi) \\
\mu(C + g)^{\mu - 1} & 0 \\
a_{23} & a_{24}
\end{bmatrix}
\begin{bmatrix}
\frac{1 - \mu}{Z + nK} \\
\theta \mu(Z + nK)\mu^{-1} \\
\sigma L^{\sigma - 1}
\end{bmatrix}
\begin{bmatrix}
e_{26} \\
\mu Y\mu^{-1} \\
a_{26}
\end{bmatrix}
\right) = 0, \tag{3.44}
\]

where
\[ e_{26}' = \frac{1 - \pi(1 - \gamma)}{C^*} [-\phi VY^\mu \mu(C + \gamma)^{\mu-1}] . \]

As explained below, singularity does occur within theoretically feasible parameter regions of this model. The boundary determined by (3.44) will be referred to as singularity-induced bifurcation boundary. As stated in Barnett and He (2008), “to the best of our knowledge, this is the first time that this type of bifurcation has been found in a macroeconometric model.”

Leeper and Sims (1994) proposed government policy control using the monetary policy rule (3.15) and tax policy rule (3.16). To investigate bifurcation of the closed-loop system under the control of government policies, Barnett and He (2008) expanded the state variable to

\[
x_c = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \\ i \\ \tau_c \end{bmatrix}.
\]

With this new augmented state vector, the linearized system (3.31)-(3.32) becomes

\[
E_1^c x_c = A_1^c x_c, \quad (3.46)
\]

\[
0 = [A_2 \ 0] x_c, \quad (3.47)
\]

where \( E_1^c \in R^{m_1^c \times m^c} = R^{6 \times 9} \), \( A_1^c \in R^{m_1^c \times m^c} = R^{6 \times 9} \), \( m_1^c = m_1 + 2 \), \( m^c = m + 2 \).

**3.5. Numerical Results**

Barnett and He (2008) applied the condition (3.44) to the closed-loop system (3.47). They first tested all pairs of parameters to determine those pairs that reach bifurcation boundaries, when the pair is varied with all other parameters set at their estimates. Pairs of parameters permitted to vary about their point estimates are allowed to take values within the intersection of their theoretically feasible ranges and the 95% confidence intervals of their estimated values. In particular, the intersection, \( \mathcal{I} \), of (3.30) and \( [\bar{p}(i) - \bar{c}\sigma_i, \bar{p}(i) + \bar{c}\sigma_i] \),

where \( \bar{p}(i) \) is the estimated value of parameter \( p(i) \), \( \sigma_i \) is the standard error of the estimate, and \( \bar{c} \) is the critical value of the 95\(^{th}\)-percentile confidence interval for \( N(0,1) \).
The estimation information for the parameters $\mu$, $g$, and $\beta$ are in Table 3.1. All estimation information is taken directly from the Leeper and Sims paper. No changes are made in their models.

### Table 3.1 Estimation of $\mu$, $g$, and $\beta$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>$\mathcal{E}$ Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>1.0248</td>
<td>0.324</td>
<td>[1, 1.6598]</td>
</tr>
<tr>
<td>$g$</td>
<td>0.0773</td>
<td>0.292</td>
<td>[0, 0.6496]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.1645</td>
<td>0.288</td>
<td>[0, 0.7290]</td>
</tr>
</tbody>
</table>

Note: Since $g$ is an exogenous variable, rather than a parameter, the “estimate” is the sample mean and the “standard error” is the sample standard deviation.

To illustrate what happens when parameter values cross the singularity boundary, consider the parameter $\beta$. Table 3.2 displays the changes of finite eigenvalues, $\lambda_1, \ldots, \lambda_8$, when $\beta$ varies.

### Table 3.2 Eigenvalue Changes

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
<th>$\lambda_6$</th>
<th>$\lambda_7$</th>
<th>$\lambda_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.080</td>
<td>0.120</td>
<td>0.160</td>
<td>0.165</td>
<td>0.170</td>
<td>0.200</td>
<td>0.240</td>
<td></td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>1.002</td>
<td>1.002</td>
<td>1.002</td>
<td>1.002</td>
<td>1.002</td>
<td>1.002</td>
<td>1.002</td>
<td></td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0.080</td>
<td>0.120</td>
<td>0.160</td>
<td>0.165</td>
<td>0.170</td>
<td>0.200</td>
<td>0.240</td>
<td></td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>-0.303</td>
<td>-0.262</td>
<td>-0.220</td>
<td>-0.215</td>
<td>-0.210</td>
<td>-0.178</td>
<td>-0.135</td>
<td></td>
</tr>
<tr>
<td>$\lambda_5$</td>
<td>-0.098</td>
<td>-0.084</td>
<td>-0.077</td>
<td>-0.076</td>
<td>-0.075</td>
<td>-0.072</td>
<td>-0.069</td>
<td></td>
</tr>
<tr>
<td>$\lambda_6$</td>
<td>-0.002</td>
<td>-0.003</td>
<td>-0.003</td>
<td>-0.003</td>
<td>-0.004</td>
<td>-0.004</td>
<td>-0.004</td>
<td></td>
</tr>
<tr>
<td>$\lambda_7$</td>
<td>3.101</td>
<td>5.177</td>
<td>8.237</td>
<td>8.682</td>
<td>9.254</td>
<td>13.416</td>
<td>28.401</td>
<td></td>
</tr>
<tr>
<td>$\lambda_8$</td>
<td>-117.790</td>
<td>-204.703</td>
<td>-1811.413</td>
<td>$\infty$</td>
<td>1456.294</td>
<td>195.888</td>
<td>58.059</td>
<td></td>
</tr>
</tbody>
</table>

The first row in Table 3.2 contains the settings of $\beta$ that Barnett and He (2008) explore. The second through the ninth rows are the corresponding finite eigenvalues of the linearized model at each setting of $\beta$. There are three more eigenvalues, which are not shown in the
table. Those eigenvalues are infinite. The table shows that when the value of $\beta$ increases and crosses the bifurcation boundary, $\lambda_8$ decreases rapidly to $-\infty$, spikes suddenly from $-\infty$ to $+\infty$, and then decreases from $+\infty$.

Table 3.2 clearly shows that the Leeper and Sims model has a structural change in dynamics, when $\beta$ crosses the singularity-induced bifurcation boundary. The two regions separated by the boundary exhibit drastically different dynamical behaviors. Very small changes in $\mu$ can cause bifurcation, independently of the setting of $g$ or $\beta$.

Also shown by Table 3.2, such singularity bifurcations can have dramatic effects. The number of dynamic equations and the number of algebraic equations change, when the singularity-induced bifurcation boundary is reached.

4. New Keynesian Model

4.1. Introduction

Recently, there has been much policy interest in New Keynesian models. As a result, Barnett and Duzhak (2008, 2010, 2013) have explored bifurcation within the class of New Keynesian models. They have studied different specifications of monetary policy rules within the New Keynesian functional structure. In initial research in this area, Barnett and He (2008) found a New Keynesian Hopf bifurcation boundary, with the setting of the policy parameters influencing the existence and location of the bifurcation boundary. Hopf bifurcation is the most commonly encountered type of bifurcation boundary found among economic models. The existence of a Hopf bifurcation boundary is accompanied by regular oscillations within a neighborhood of the bifurcations boundary. Following a more extensive and systemic search of the parameter space, Barnett and Duzhak (2010) also found the existence of Period Doubling (flip) bifurcation boundaries in the same class of models. Central results needed in this research are the theorems above on the existence and location of the Hopf bifurcation boundaries in each of the cases that considered.

The usual New Keynesian log-linearized model consists of a forward-looking IS-curve, describing consumption smoothing behavior, and a New Keynesian Phillips curve, derived from price optimization by monopolistically competitive firms in the presence of nominal rigidities. The third equation is a monetary policy rule. Monetary policy rules include different specifications of Taylor’s rule and inflation targeting. Barnett and Duzhak (2010) use eigenvalues of the linearized system of difference equations to locate Hopf bifurcation boundaries. They also investigate the effects of different monetary policy rules on bifurcation boundary locations. In each case, they solve numerically for the location and properties of the bifurcation boundary and its dependency upon policy rule parameter settings. They use two
types of New Keynesian models. One type can be reduced to produce a $2 \times 2$ Jacobian. The other type produces a $3 \times 3$ Jacobian. To our knowledge there is no theoretical literature for the $3 \times 3$ case in the economics literature. In the $3 \times 3$ case Barnett and Duzhak (2010) employ the theorem on Hopf bifurcation from the engineering literature.

Beginning with Grandmont’s findings with a classical model, we continue in this survey to follow the path from the Bergstrom-Wymer policy-relevant Keynesian model, then to the Euler equation macroeconometric models, and now to New Keynesian models.

4.2. The Model

The main assumption of New Keynesian economic theory is that there are nominal price rigidities preventing prices from adjusting immediately and thereby creating disequilibrium unemployment. Price stickiness is often introduced in the manner proposed by Calvo (1983). The model below, used as the theoretical background for Barnett and Duzhak’s (2008, 2010, 2013) log linearized bifurcation analysis, is based closely upon Walsh (2003), section 5.4.1, pp. 232-239, which in turn is based upon the monopolistic competition model of Dixit and Stiglitz (1977).

The model consists of consumers, firms, and monetary policy authority. Consumers derive utility from the composite consumption good, $C_t$, real money balances, and leisure. Consumers supply their labor in a competitive labor market and receive labor income, $w_t N_t$. Consumers own the firms producing consumption goods and receive all profits, $\pi_t$. The representative consumer can allocate wealth to money and bonds and choose the aggregate consumption stream by solving the utility maximization problem.

Firms operate in a monopolistically competitive market. Each firm has pricing power over the goods it sells. Price rigidity by the firm results from the fact that a random fraction of firms does not adjust its product price in each period. The remaining firms adjust prices to their optimal levels. Firms make their production and price-setting decisions by solving the cost minimization and pricing decision problems, such that

$$x_t = E_t x_{t+1} - \frac{\ell_t - E_t \pi_{t+1}}{\sigma},$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t,$$

where $\pi_t$ is the inflation rate at time $t$; $\ell_t$ is the interest rate; $x_t = (\hat{y}_t - \hat{y}_t^f)$ is the gap between actual output percentage deviation $\hat{y}_t$ and the flexible-price output percentage deviation $\hat{y}_t^f$; $\sigma$ is a degree of relative risk aversion; $E_t$ is the expectations operator conditionally upon information at time $t$, and $\beta$ is the discount factor. The first equation (4.1) provides the demand side of the economy and is a forward-looking IS curve that relates the
output gap to the real interest rate. Equation (4.2) is the New-Keynesian Phillips curve, which represents the supply side by describing how inflation is driven by the output gap and expected inflation. The resulting system of two equations has three unknown variables: inflation, output gap, and nominal interest rate.

Barnett and Duzhak’s (2008, 2010, 2013) need one more equation to close the model. The remaining necessary equations will be a monetary policy rule, in which the central bank uses a nominal interest rate as the policy instrument. Numerous types of monetary policy rules have been discussed in the economic literature. Two main policy classes are targeting rules and instrument rules. A simple instrument rule relates the interest rate to a few observable variables. The most famous such rule is Taylor’s rule. Taylor demonstrated that a simple reaction function, with a short-term interest-rate policy instrument, responding to inflation and output gap, follows closely the observed path of the Federal Funds rate. Researchers have tried to modify Taylor’s rule to acquire a better fit to the data. Barnett and Duzhak (2010) initially centered analysis on specification of the current-looking Taylor rule, then on forward-looking, backward-looking and hybrid Taylor rules. Recent literature also proposes many ways to define an inflation target. Barnett and Duzhak (2010) considered current-looking, forward-looking and backward-looking inflation target.

4.3. Determinacy and Stability Analysis

There are some mathematics results that are useful in finding bifurcation boundaries.

**Lemma 4.1** For a matrix \( \mathbf{A} = [a_{ij}] \), with \( i, j = 1, 2, 3 \), a pair of complex conjugate eigenvalues lies on the unit circle and another eigenvalue lies inside the unit circle, if and only if

\[
\begin{align*}
(a) & \ |x| < 1, \\
(b) & \ |x + z| < 1 + y, \\
(c) & \ y - xz = 1 - x^2,
\end{align*}
\]

where \( z, y, \) and \( x \) are the coefficients of the characteristic equation \( \lambda^3 + z\lambda^2 + y\lambda + x = 0 \) of the matrix \( \mathbf{A} \).

**Proof.** See Barnett and Duzhak (2010), Lemma 3.1.

**Theorem 4.1 (Existence of Hopf Bifurcation in 3 Dimensions)**

Consider a map \( \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \phi) \), where \( \mathbf{x} \) has 3 dimensions. Let \( \mathbf{J} \) be the Jacobian of the transformation, and let the characteristic polynomial of the Jacobian be \( P(\lambda) = \lambda^3 + z\lambda^2 + y\lambda + x = 0 \). Assume that for one of the equilibria \( (\mathbf{x}^*, \phi^*) \), there is a critical value \( \phi^c_i \) for one of
the parameters, \( \varphi_i^* \), in \( \varphi^* \) such that eigenvalue conditions (a),(b) and (c) and transversality condition (d) hold, where:

\[
\begin{align*}
(a) & \quad |x| < 1, \\
(b) & \quad |x + z| < 1 + y, \\
(c) & \quad y - xz = 1 - x^2, \\
(d) & \quad \frac{\partial |\lambda_j(x^*, \varphi^*)|}{\partial \varphi_i^*} \bigg|_{\varphi_i^* = \varphi_i^c} \neq 0 \text{ for the complex conjugates with } j = 1, 2.
\end{align*}
\]

Then there is an invariant closed curve Hopf-bifurcating from \( \varphi^* \).

**Proof.** See Barnett and Duzhak (2010), Theorem 3.2.

i. **Current-Looking Taylor Rule**

The current-looking Taylor rule is:

\[
i_t = a_1 \pi_t + a_2 x_t,
\]

where \( a_1 \) is the coefficient of the central bank’s reaction to inflation and \( a_2 \) is the coefficient of the central bank’s reaction to the output gap. Barnett and Duzhak (2010) consider the forward-looking, backward-looking, and the hybrid Taylor rules.

They use these three equations:

\[
\begin{align*}
x_t &= E_t x_{t+1} - \frac{i_t - E_t \pi_{t+1}}{\sigma}, \\
\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t, \\
i_t &= a_1 \pi_t + a_2 x_t.
\end{align*}
\]

This 3-equation system constitutes a New Keynesian model.

In order to analyze the model’s determinacy and stability properties, Barnett and Duzhak (2010) display it in the standard form:

\[
E_t x_{t+1} = C x_t + \delta_t,
\]

where \( \delta_t \) is a vector of disturbances and \( E_t \) is the expectations operator conditional upon information in period \( t \). With the current-looking Taylor rule, they use the following version, which is not in closed form:

\[
A E_t x_{t+1} = B x_t + \delta_t,
\]
where

\[
A = \begin{bmatrix}
1 & \frac{1}{\sigma} & 0 \\
0 & \beta & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & \frac{1}{\sigma} \\
-\kappa & 1 & 0 \\
\frac{a_2}{a_1} & 0 & -1 \\
\end{bmatrix}, \quad x_t = \begin{bmatrix}
x_t \\
\pi_t \\
i_t \\
\end{bmatrix}.
\]

Obtaining the matrix \( C = A^{-1}B \) is impossible, since \( A \) is a singular matrix.

Therefore, they reduce the system of three equations to a system of two log-linearized equations by substituting Taylor’s rule into the consumption Euler equation. The resulting system of expected difference equations has a determinate solution, if the number of eigenvalues outside the unit circle equals the number of forward-looking variables (Blanchard and Kahn (1980)). That system of two equations has the following form:

\[
\begin{bmatrix}
1 & \frac{1}{\sigma} \\
0 & \beta \\
\end{bmatrix} \begin{bmatrix}
E_t x_{t+1} \\
E_t \pi_{t+1} \\
\end{bmatrix} = \begin{bmatrix}
1 + \frac{a_2}{\sigma} & -\frac{a_1}{\sigma} \\
-\kappa & 1 \\
\end{bmatrix} \begin{bmatrix}
x_t \\
\pi_t \\
\end{bmatrix},
\]

which can be written as

\[
A E_t x_{t+1} = B x_t,
\]

where

\[
x_t = \begin{bmatrix}
x_t \\
\pi_t \\
\end{bmatrix}, \quad A = \begin{bmatrix}
1 & \frac{1}{\sigma} \\
0 & \beta \\
\end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix}
1 + \frac{a_2}{\sigma} & -\frac{a_1}{\sigma} \\
-\kappa & 1 \\
\end{bmatrix}.
\]

Premultiply the system by the inverse matrix \( A^{-1} \),

\[
A^{-1} = \begin{bmatrix}
1 & -\frac{1}{\beta \sigma} \\
0 & \frac{1}{\beta} \\
\end{bmatrix},
\]

results in

\[
E_t x_{t+1} = C x_t \quad \text{or} \quad \begin{bmatrix}
E_t x_{t+1} \\
E_t \pi_{t+1} \\
\end{bmatrix} = \begin{bmatrix}
1 + \frac{a_2 \beta + \kappa}{\sigma \beta} & -\frac{a_1 \beta - 1}{\sigma \beta} \\
-\kappa & \frac{1}{\beta} \\
\frac{1}{\beta} & 1 \\
\end{bmatrix} \begin{bmatrix}
x_t \\
\pi_t \\
\end{bmatrix},
\]

where \( C = A^{-1}B \).

The uniqueness and stability requires both eigenvalues to be outside the unit circle. The eigenvalues of \( C \) are the roots of the characteristic polynomial.
\[ p(\lambda) = \det(C - \lambda I) = \lambda^2 - \lambda \left[ 1 + \frac{a_2 \beta + k}{\sigma \beta} + \frac{1}{\beta} \right] + \frac{\sigma \beta + a_2 \beta + ka_1 \beta}{\sigma \beta^2} \]

Define \( D \) as

\[ D = \left[ 1 + \frac{a_2 \beta + k}{\sigma \beta} + \frac{1}{\beta} \right]^2 - 4 \frac{\sigma \beta + a_2 \beta + ka_1 \beta}{\sigma \beta^2} \]

Then the eigenvalues are

\[ \lambda_1 = \frac{1}{2} \left( 1 + \frac{a_2 \beta + k}{\sigma \beta} + \frac{1}{\beta} + \sqrt{D} \right) \]

and

\[ \lambda_1 = \frac{1}{2} \left( 1 + \frac{a_2 \beta + k}{\sigma \beta} + \frac{1}{\beta} - \sqrt{D} \right) \]

It can be shown that both eigenvalues will be outside the unit circle, if and only if

\[ (a_1 - 1)\kappa + (1 - \beta)a_2 > 0. \tag{4.4} \]

If \( a_1 > 1 \), (4.4) holds. Interest rate rules that meet this criterion are called active. This relationship is also known as Taylor’s principle, which prescribes that the interest rate should be set higher than the increase in inflation. Monetary policy satisfying the Taylor’s principle is thought to eliminate equilibrium multiplicities.

In this case, the Jacobian of the New Keynesian model can be written in the form:

\[ J = \begin{bmatrix} 1 + \frac{a_2 \beta + \kappa}{\sigma \beta} & \frac{a_1 \beta - 1}{\sigma \beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \]

The goal is to apply the Theorem 1.1 (Hopf bifurcation existence theorem) to the Jacobian of the model. Before doing that, a bifurcation parameter needs to be selected to vary, while holding other parameters constant. The model is parameterized by:

\[ \varphi = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \end{pmatrix} \]
Candidates for a bifurcation parameter are coefficients for the monetary policy rule, $a_1$ and $a_2$. For a New Keynesian model with current looking Taylor rule, the following result is proved in Barnett and Duzhak (2008):

**Proposition 4.1** The new Keynesian model with current-looking Taylor rule ((4.1),(4.2),(4.3)) undergoes a Hopf bifurcation if and only if the discriminant of the characteristic equation is negative and $a_2^c = \sigma \beta - \kappa a_1 - \sigma$.

Combing the critical value for $a_2$ with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary, the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{\sigma + \sigma \beta - \kappa a_1 \beta + \kappa}{\sigma \beta^2} < 1.$$  

ii. **Forward-Looking Taylor Rule**

A forward-looking Taylor rule sets the interest rate according to expected future inflation rate and output gap, in accordance with the following equation:

$$i_t = a_1 E_t \pi_{t+1} + a_2 E_t \chi_{t+1}. \quad (4.5)$$

The model is again parameterized by

$$\varphi = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \end{pmatrix}.$$  

The resulting Jacobian has the following form:

$$J = \begin{bmatrix} \frac{\sigma}{\sigma - a_2} + \frac{\kappa(1-a_1)}{(\sigma - a_2)\beta} & \frac{a_1 - 1}{(\sigma - a_2)\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$  

Candidates for a bifurcation parameter are coefficients, $a_1$ and $a_2$, for the monetary policy rule. The following result was proved in Barnett and Duzhak (2008):

**Proposition 4.2** The new Keynesian model with forward-looking Taylor rule ((4.1), (4.2),(4.5)) undergoes a Hopf bifurcation if and only if the discriminant of the characteristic equation is negative and $a_2^c = -\frac{\sigma}{\beta} + \sigma$. 
Combing the critical value for $a_2$ with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary, the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-1 < \frac{1}{2} \left( \beta + \frac{\kappa (1 - a_1)}{\sigma} + \frac{1}{\beta} \right) < 1.$$

In order to analyze the forward-looking Taylor rule numerically for the possibility of period doubling bifurcation, an algorithm is needed. An algorithm for numerical bifurcation analysis is based on the following technique:

Given the $i^{th}$ iterate of the fixed point, $f^i(x) - x = 0$, a period doubling bifurcation is detected according to the following test function:

$$\varphi_{PD} = \det(J^{(i)} + I_n),$$

where $J^{(i)}$ is the Jacobian matrix of the iterated map $f^i$. A period-doubling bifurcation will occur whenever $\varphi_{PD} = 0$.

Location of the period-doubling bifurcation boundary can be implemented using the software continuation package CONTENT, which is developed by Yuri Kuznetsov and V.V. Levitin. To locate a bifurcation boundary, Barnett and Duzhak (2010) select parameter $a_2$ to be a free parameter for numerical bifurcation analysis of the new Keynesian model with forward-looking Taylor’s rule. A period-doubling bifurcation point is found at $a_2 = 2.994$ with the other parameters set as in Barnett and Duzhak (2010). The nature of the state space solution will depend upon the bifurcation boundary’s side, on which the parameters are located. If parameter $a_2$ is moved to 3 with the other parameters set as in Barnett and Duzhak’s (2010 appendix table, the solution becomes periodic.

Period-doubling bifurcation can arise when the central banker reacts aggressively to the expected future values of the output gap. Along the bifurcation boundary, the values of parameter $a_2$ are within the range of 2.75 and 3. At those settings, the central banker actively reacts to the expected future values of inflation and even more aggressively to the forecast based values of the output gap in the forward-looking Taylor rule.

### iii. Hybrid Taylor Rule

Consider the Taylor rule of the following form,

$$i_t = a_1 E_t \pi_{t+1} + a_2 x_t$$

(4.6)
Substituting (4.6) into consumption Euler equation, Barnett and Duzhak (2008) acquire the Jacobian:

\[
\mathbf{J} = \begin{bmatrix}
1 + \frac{a_2}{\sigma} + \frac{\kappa(1-a_1)}{\sigma \beta} & \frac{a_1 - 1}{\sigma \beta} \\
-\frac{\kappa}{\beta} & \frac{1}{\beta}
\end{bmatrix}.
\]

Candidates for a bifurcation parameter are coefficients, \(a_1\) and \(a_2\), for the monetary policy rule. The following result was proved in Barnett and Duzhak (2008):

**Proposition 4.3** The new Keynesian model with Hybrid-Taylor rule, equations (4.1),(4.2),(4.6) undergoes a Hopf bifurcation, if and only if the discriminant of the characteristic equation is negative and \(a_2^c = \beta \sigma - \sigma\).

Combining the critical value for \(a_2\) with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary, the bifurcation boundary is the set of parameter values satisfying the following condition:

\[-1 < \frac{\sigma(1 + \beta^2) + \kappa(1-a_1)}{2\sigma \beta} < 1.\]

iv. **Current-Looking Inflation Targeting**

The inflation targeting equation

\[
i_t = a_1 \pi_t
\]

(4.7)
can be used instead of the Taylor rule. Then the Jacobian is

\[
\mathbf{J} = \begin{bmatrix}
\frac{\sigma \beta + \kappa}{\sigma \beta} & \frac{a_3 \beta - 1}{\sigma \beta} \\
-\frac{\kappa}{\beta} & \frac{1}{\beta}
\end{bmatrix}.
\]

The model is characterized by

\[
\varphi = \begin{bmatrix}
\beta \\
\sigma \\
\kappa \\
a_1
\end{bmatrix}.
\]

A candidate for a bifurcation parameter is the coefficient, \(a_1\), for the monetary policy rule. The following result is proved in Barnett and Duzhak (2008):
Proposition 4.4  The new Keynesian model with current-looking inflation targeting, ((4.1),(4.2),(4.7)), produces a Hopf bifurcation, if and only if the discriminant of the characteristic equation is negative and $a_1^c = \frac{\sigma \beta - \sigma}{\kappa}$.

Combing the critical value for $a_1$ with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary, the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-3 < \frac{\sigma + \kappa}{\sigma \beta} < 1.$$  

v.  Forward-Looking Inflation Targeting

The following is the forward-looking inflation targeting rule,

$$i_t = a_1 E_t \pi_{t+1}.$$  \hspace{1cm} (4.8)

Then the Jacobian is

$$J = \begin{bmatrix} 1 + \frac{\kappa (1-a_1)}{\sigma \beta} & \frac{a_1 - 1}{\sigma \beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$  

The model is parameterized by

$$\varphi = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \end{pmatrix}.$$  

The following proposition about the forward-looking inflation-targeting New Keynesian model is proved in Barnett and Duzhak (2008):

Proposition 4.5  The new Keynesian model with forward-looking inflation targeting, ((4.1),(4.2), (4.8)), produces a Hopf bifurcation, if and only if the discriminant of the characteristic equation is negative and $\beta^c = 1$.

Combing the critical value for $\beta$ with the condition on the discriminant of the characteristic polynomial to provide the condition defining the Hopf bifurcation boundary, the bifurcation boundary is the set of parameter values satisfying the following condition:

$$-3 < \frac{\kappa (a_1 - 1)}{2 \sigma} < 1.$$
Parameter $\beta$ is the discount factor from the representative agent’s optimization problem. It is also a coefficient in the Phillips curve scaling the impact of expected inflation. Some authors assume for simplicity that $\beta = 1$. Surprisingly, this result does not require separate setting of $a_1$ to attain Hopf bifurcation. Under the conditions of this proposition, no freedom remains to select $a_1$ independently. This conclusion is conditional upon the assumption that the log-linearized New Keynesian model is a good approximation to the economy and that the discriminant of the characteristic equation is negative. In such cases, setting the discount factor $\beta$ equal to unity is not appropriate.

If the model is parameterized by discount factor $\beta = 0.98$, then the dynamic solution in phase space (inflation rate plotted against output gap) will be periodic. It is interesting to see what happens to that solution path, if the parameter value is located directly on the bifurcation boundary. In that case, the solution in phase space will become an invariant limit cycle.

vi. Backward-Looking Taylor Rule

Backward-looking monetary policy rules are intended to prevent expectations driven fluctuations. Carlstrom and Fuerst (2000) argued for a backward-looking interest rate rule reacting aggressively to past values of inflation. Such a policy should be sufficient for determinacy of equilibria. Similar results can be found in Eusepi (2005).

With a backward-looking Taylor rule, the central bank sets an interest rate according to the past values of inflation and output gap as follows:

$$i_t = a_1 \pi_{t-1} + a_2 x_{t-1}.$$  
(4.9)

The New Keynesian model with backward-looking Taylor rule produces the following system:

$$E_t x_{t+1} = C x_t,$$

where

$$C = \begin{bmatrix} 1 + \frac{\kappa}{\sigma \beta} & -\frac{1}{\sigma \beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & 1 & 0 \\ a_2 & a_1 & 0 \end{bmatrix}, \quad x_t = \begin{bmatrix} \pi_t \\ x_t \\ i_t \end{bmatrix}.$$

Matrix $C$ has the characteristic polynomial

$$p(\lambda) = \det(C - \lambda I) = \lambda^3 - \frac{\sigma(1 + \beta)}{\sigma \beta} \lambda^2 + \frac{\sigma - \beta a_2}{\sigma \beta} \lambda + \frac{\kappa a_1 + a_2}{\sigma \beta}.$$

The following proposition is proved in Barnett and Duzhak (2010).
**Proposition 4.6** The New Keynesian model with backward-looking Taylor rule produces a Hopf bifurcation, if the transversality condition, \[ \frac{\partial \lambda_j(x^*, \varphi^*)}{\partial \varphi_i^*} \bigg|_{\varphi_i^* = \varphi_i^*} \neq 0, \] holds, and if the parameters \( \alpha_1 \) and \( \alpha_2 \) satisfy the following three conditions at the equilibrium:

\[
\begin{align*}
(a) \quad & \left| \frac{a_2 + \kappa a_1}{\sigma \beta} \right| < 1, \\
(b) \quad & (a_2 - 2\sigma)(1 + \beta) + \kappa(a_1 - 1) < 0, \\
(c) \quad & \frac{\sigma - \beta a_2}{\sigma \beta} + \frac{(\kappa a_1 + a_2)(\sigma(1+\beta) + \kappa)}{\sigma^2 \beta^2} = 1 - \left( \frac{\kappa a_1 + a_2}{\sigma \beta} \right)^2.
\end{align*}
\]

**vi. Backward-Looking Inflation Targeting**

Backward-looking inflation targeting sets the interest rate according to past values of inflation, as follows:

\[ i_t = a_1 \pi_{t-1}. \quad (4.10) \]

The Jacobian is

\[
J = \begin{bmatrix}
1 + \frac{\kappa}{\sigma \beta} & -\frac{1}{\sigma \beta} & \frac{1}{\sigma} \\
-\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\
0 & \frac{1}{a_1} & 0
\end{bmatrix}
\]

The Jacobian has the characteristic polynomial

\[ p(\lambda) = \lambda^3 - \frac{\sigma(1 + \beta) + \kappa}{\sigma \beta} \lambda^2 + \frac{1}{\beta} \lambda + \frac{\kappa a_1}{\sigma \beta}. \]

The following proposition is proved in Barnett and Duzhak (2010).

**Proposition 4.7** The New Keynesian model with backward-looking inflation targeting produces a Hopf bifurcation, if the transversality condition, \[ \frac{\partial \lambda_j(x^*, \varphi^*)}{\partial \varphi_i^*} \bigg|_{\varphi_i^* = \varphi_i^*} \neq 0, \] holds, and if the parameters \( \varphi_i^* \) satisfy the following three conditions at the equilibrium:

\[
\begin{align*}
(a) \quad & \left| \frac{\kappa a_1}{\sigma \beta} \right| < 1, \\
(b) \quad & a_1 > 1, \\
(c) \quad & \frac{\sigma^2 \beta + \kappa a_1(\sigma(1+\beta) + \kappa)}{\sigma^2 \beta^2} = 1 - \left( \frac{\kappa a_1}{\sigma \beta} \right)^2.
\end{align*}
\]
vii. Current-Looking Taylor Rule with Interest Rate Smoothing Term

If the interest rate path is computed from the optimal interest rate rule, it will be much more volatile than the path observed in practice. Central bankers prefer to avoid volatility in interest rates. To smooth interest rates in models, some economists include a lagged interest rate term in the interest rate rule, as follows:

\[ i_t = (1 - a_3)(a_1 \pi_t + a_2 x_t) + a_3 i_{t-1}. \]  

(4.11)

The model is parameterized by

\[ \varphi = \begin{pmatrix} \beta \\ \sigma \\ \kappa \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}. \]

Parameter \( a_3 \) describes the degree of interest rate smoothing by the central bank and is assumed to be between zero and one.

The new Keynesian model with current-looking Taylor rule and interest rate smoothing has the following matrix form:

\[ E_t x_{t+1} = C x_t, \]

where

\[ C = \begin{bmatrix} 1 + \frac{\kappa}{\sigma \beta} & -\frac{1}{\sigma \beta} & \frac{1}{\sigma} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ -a_2(a_3 - 1) + \frac{(-1 + a_3)(a_1 \sigma - a_2) \kappa}{\sigma \beta} & -\frac{(-1 + a_3)(a_1 \sigma - a_2)}{\sigma \beta} & -\frac{a_2(-1 + a_3)}{\sigma} + a_3 \end{bmatrix} \]

and

\[ x_t = \begin{bmatrix} x_t \\ \pi_t \\ i_t \end{bmatrix}. \]

This system has the following characteristic polynomial:

\[ p(\lambda) = \lambda^3 + \left( \frac{a_2(a_3 - 1)}{\sigma} - 1 - a_3 - \frac{\kappa}{\sigma \beta} - \frac{1}{\beta} \right) \lambda^2 + \left( \frac{\kappa a_1 - a_2 a_3 + a_2 + \kappa a_3 (1 - a_1)}{\sigma \beta} + a_3 + \frac{1 + a_3}{\beta} \right) \lambda - \frac{a_3}{\beta} \]
The following proposition is proved in Barnett and Duzhak (2010).

**Proposition 4.8** The New Keynesian model consisting of ((4.1), (4.2), (4.11)) produces a Hopf bifurcation, if the transversality condition 
\[ \frac{\partial |\lambda(x, \varphi)|}{\partial \varphi_i} \Bigg|_{\varphi_i^*=\varphi_i^c} \neq 0 \]
holds, and if the parameters \( \varphi_i^* \) satisfy the following three conditions at the equilibrium:

(a) \( a_3 - \beta < 0 \),
(b) \( a_1 > 1 \),
(c) \( \frac{1-a_2^2}{\beta} - (1 - a_3) + \frac{a_3(a_3-1)}{\beta^2} + \frac{a_3a_2(a_3-2)+\kappa a_1(1-a_3)+a_2+a_3}{\sigma \beta} = 0 \).

viii. **Backward-Looking Taylor Rule With Interest Rate Smoothing Term**

Backward-looking rules are commonly viewed to be the least prone to indeterminacy, and thereby commonly advocated for use by monetary authorities. The backward-looking Taylor rule with interest rate smoothing is specified as follows:

\[ i_t = (1 - a_3)(a_1 \pi_{t-1} + a_2 x_{t-1}) + a_3 i_{t-1}. \] (4.12)

The Jacobian is

\[
\begin{bmatrix}
1 + \frac{\kappa}{\sigma \beta} & -\frac{1}{\sigma \beta} & \frac{1}{\sigma} \\
-\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\
a_2(1 - a_3) & a_1(1 - a_3) & a_3
\end{bmatrix}
\]

with characteristic polynomial

\[
p(\lambda) = \lambda^3 - \left(1 + a_3 + \frac{\kappa}{\sigma \beta} + \frac{1}{\beta}\right) \lambda^2 + \left(\frac{a_2 \beta(a_3 - 1) + \kappa a_3 + \sigma (1 + a_3)}{\sigma \beta} + a_3\right) \lambda
\]

\[+\frac{\kappa a_1(1-a_3)+a_2(1-a_3)-\sigma a_3}{\sigma \beta}.
\]

The following proposition is proved in Barnett and Duzhak (2010).

**Proposition 4.9.** The New Keynesian model consisting of ((4.1), (4.2), (4.12)) produces a Hopf bifurcation, if the transversality condition 
\[ \frac{\partial |\lambda(x, \varphi)|}{\partial \varphi_i} \Bigg|_{\varphi_i^*=\varphi_i^c} \neq 0 \]
holds, and if the parameters \( \varphi_i^* \) satisfy the following three conditions at the equilibrium:
\[
\left( a \right) \left| \frac{\kappa a_1(1-a_3)+a_2(1-a_3)-\sigma a_3}{\sigma \beta} \right| < 1,
\]

\[
\left( b \right) \left| \frac{\kappa a_1(1-a_3)+a_2(1-a_3)-\sigma a_3-\kappa-\sigma}{\sigma \beta} - 1 - a_3 \right| < \frac{a_2 \beta(\kappa a_3+\sigma(1+a_3)) + a_3}{\sigma \beta},
\]

\[
\left( c \right) \frac{a_2 \beta(\kappa a_3+\sigma(1+a_3)) + a_3 + [(a_2+\kappa a_1)(1-a_3)\sigma a_3][\sigma \beta(1+a_3)+\kappa+\sigma]}{(\sigma \beta)^2} = 1 - \left( \frac{(a_2+\kappa a_1)(1-a_3)\sigma a_3}{\sigma \beta} \right)^2.
\]

The existence of a period-doubling bifurcation is found in this case through numerical procedures. Here \( a_2 \) is the parameter to vary, while other parameters are fixed. The first period doubling bifurcation point is found at \( a_2 = 5.7 \). Period doubling bifurcation will occur for large values of the parameter \( a_2 \). In short, aggressive reaction of the central bank to past values of the output gap can lead to a period doubling bifurcation within this model.

If parameters \( a_2 \) and \( a_1 \) are varied while holding the other parameters constant, a numerical solution for this boundary was obtained with the starting point \( a_2 = 5.7 \), and then by simultaneously varying parameters \( a_2 \) and \( a_1 \). The located period doubling bifurcation boundary has values of the parameter \( a_2 \) within the range from 5.98 to 6.02. Hence, a period doubling bifurcation will occur within a very narrow set of parameters \( a_2 \) around 6. These results were acquired relative to the standard calibration in the appendix in Barnett and Duzhak (2010). Changing the interest rate smoothing parameter \( a_3 \) leads to a different critical period-doubling bifurcation value for the parameter \( a_2 \).

ix. **Hybrid Rule With Interest Rate Smoothing Term**

The hybrid rule with interest rate smoothing is proposed in Clarida, Gali and Gertler (1998). This rule specification is widely believed to match the empirics of the monetary policy for the main countries of the European Union, Japan, and the United States. Based on this rule, the central banker sets a short-term interest rate in accordance with a forecast-based value of inflation, the current value of the output gap, and a past value of the interest rate, as follows:

\[
i_t = (1 - a_3)(a_1 \pi_{t+1} + a_2 x_t) + a_3 i_{t-1}. \tag{4.13}
\]

The New Keynesian model consisting of equation ((4.1), (4.2), (4.13)) can be represented as follows:

\[
A E_t x_{t+1} = B x_t,
\]

where
\[
A = \begin{bmatrix}
1 & \frac{1}{\sigma} & 0 \\
0 & \beta & 0 \\
0 & -a_1(1-a_3) & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & \frac{1}{\sigma} \\
-\kappa & 1 & 0 \\
a_2(1-a_3) & 0 & a_3
\end{bmatrix}, \quad x_t = \begin{bmatrix}
x_t \\
\pi_t \\
i_{t-1}
\end{bmatrix}.
\]

This model has the following Jacobian:

\[
J = \begin{bmatrix}
1 + \frac{\kappa}{\sigma\beta} & -\frac{1}{\sigma} & \frac{1}{\sigma} \\
-\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\
a_1(1-a_3) + a_2(1-a_3) & a_1(1-a_3) & a_3
\end{bmatrix}
\]

The characteristic polynomial is given by:

\[
p(\lambda) = \lambda^3 \left(1 + a_3 + \frac{\kappa}{\sigma\beta} + \frac{1}{\beta}\right)\lambda^2 + \left(a_3 + \frac{1 + a_3}{\beta} - \frac{a_2(1-a_3)}{\sigma} + \frac{a_3\kappa + a_1\kappa(1-a_3)}{\sigma\beta}\right)\lambda
\]

\[
-\frac{a_3}{\beta} - \frac{a_3a_2}{\sigma\beta} + \frac{a_2}{\sigma\beta}
\]

The following proposition is proved in Barnett and Duzhak (2010).

**Proposition 4.10** The New Keynesian model consisting of ((4.1), (4.2),(4.13)) produces a Hopf bifurcation, if the transversality condition \(\frac{\partial J(x^*, \phi^*)}{\partial \phi^*_i} \bigg|_{\phi^* = \phi^*_i = 0} \neq 0\) holds, and if the parameters \(\phi^*_i\) satisfy the following three conditions at the equilibrium:

(a) \(\left|\frac{a_3}{\beta} - \frac{a_3a_2}{\sigma\beta} + \frac{a_2}{\sigma\beta}\right| < 1\),

(b) \(\left|\frac{a_2(1-a_3)\kappa}{\sigma\beta} - 1 - a_3 - \frac{1}{\beta} - \frac{a_3}{\beta}\right| < 1 + a_3 + \frac{1 + a_3}{\beta} - \frac{a_2(1-a_3)}{\sigma} + \frac{a_3\kappa + a_1\kappa(1-a_3)}{\sigma\beta}\),

(c) \(a_3 + \frac{1 + a_3}{\beta} - \frac{a_2(1-a_3)}{\sigma} + \frac{a_2(1-a_3)\kappa}{\sigma\beta} + \left(-\frac{a_3}{\beta} + \frac{a_2(1-a_3)}{\sigma\beta}\right)\left(1 + a_3 + \frac{1}{\beta} + \frac{\kappa}{\beta\sigma}\right)\)

\[= 1 - \left(-\frac{a_3}{\beta} + \frac{a_2(1-a_3)}{\sigma\beta}\right)^2.\]

Numerical analysis of this dynamic system reveals period doubling bifurcation as follows. First vary parameter \(a_2\), while holding other parameters constant. Assuming the standard calibration in the appendix in Barnett and Duzhak (2010), the critical value of parameter \(a_2\) is 3.03. Starting with this initial condition, continue numerically searching for sections of the period doubling bifurcation boundary, first for parameters \(a_2\) and \(a_3\) with the other parameters held constant, and then for parameters \(a_2\) and \(a_1\) with the other parameters held constant.
The quality of solution for New Keynesian model with a hybrid type of Taylor rule with interest rate smoothing would change if central banker would start actively reacting to an output gap, with the policy parameter $a_2 > 3$. While simultaneously varying parameters $a_2$ and $a_3$, Barnett and Duzhak (2010) found a fold flip bifurcation point at $a_2 = 3.03$ and $a_2 = 0.46$. At this point one of the eigenvalues equals -1 and another equals 1. Once the eigenvalue crosses through this point, the solution becomes highly unstable, since all of the eigenvalues become greater or equal to 1 in absolute value.

Within the period-doubling bifurcation boundary section for parameters $a_2$ and $a_1$, with the other parameters held constant, parameter $a_2$ is located mostly between 3 and 3.15, regardless of the values of parameter $a_1$. Therefore, given the standard calibration, period doubling bifurcation will occur if the central banker actively reacts to the output gap.

Bifurcation analysis of monetary policy rules with interest rate smoothing reveal two types of bifurcation. The analytical analysis established existence of Hopf bifurcation. The numerical procedure located the possibility of period-doubling bifurcation. Furthermore, period doubling bifurcation was found for several types of policy rules using interest rate smoothing. Even backward-looking interest rate rules show evidence of this type of bifurcation, although previously thought to be the least prone to any kind of bifurcations.

5. New Keynesian Model With Regime Switching

5.1. Background

Monetary policy has seen major changes over the past decades. The 1970s were plagued by high inflation along with slow economic growth while the Central bank stayed relatively passive in its actions. With the appointment of Volcker, the Federal Reserve shifted to a more active regime which helped to combat high inflation rates present at the start of the 1980s. The following period of moderate inflation along with stable economic growth has been coined the Great Moderation. With the Great Recession following the financial crises starting in 2007, the Fed had to move aggressively.

Section 5, based on Barnett and Duzhak (2014), investigates whether bifurcations can result from regime switching over time. The parameter space of the standard New Keynesian model has been shown to be stratified into bifurcation subsets by Barnett and Duzhak (2008) and Barnett and Duzhak (2010)). The original New Keynesian model has been developed into an important tool for monetary policy (see Gali and Gertler (1999), Bernanke, Laubach, Mishkin and Posen (1999), and Leeper and Sims (1994)). Andrews (1993) and Evans (1985) study monetary policy with parameter instability. Davig and Leeper (2006) and Farmer, Waggoner, and Zha (2007) study determinacy when the Taylor rule is generalized to allow for regime
switching. There is a literature on methods to determine parameter instability in time series (see Hansen (1992) and Nyblom (1989)). Economic models of regime switching had been investigated previously in different contexts, such as Hamilton (1989) and Warne (2000). Clarida, Gali and Gertler (1999), Sims and Zha (2006), and Groen and Mumtaz (2008) find empirical support for regime switches in monetary policy.

5.2. Introduction

Barnett and Duzhak (2014) studies the dynamical behavior of standard macroeconomic models, when the monetary policy regimes can switch over time. More specifically, the policy regime is assumed to follow a Markov chain with a fixed transition matrix. The solution to the model evolved differently depending on the state of the regime. Since the standard New Keynesian model displays bifurcation in certain regions of the parameter space, the regime switching model could visit this parameter combination for one of the policy regimes, while the occasional switch to another policy regime could stabilize the solution.

The monetary authority sets the nominal interest rate as a function of current inflation. However, the response coefficient varies depending on the policy regime present at the time. The Fisher equation links the nominal interest rate to future inflation, and the real interest rate provides the second relationship. Combining the two, Barnett and Duzhak (2014) got an equation that relates future inflation to current inflation and the real interest rate. Taking the latter as given, a system of two linear difference equations is acquired for inflation in the two regimes.

To perform bifurcation analysis, the matrix that governs the evolution of current inflation to future inflation is used. The relevant properties of this matrix are the sign and magnitude of the eigenvalues. The procedure is to set up and solve the characteristic polynomial. The solution demonstrates two main findings with respect to bifurcations. First, for the basic setup, Barnett and Duzhak (2014) find no possibility of a Hopf bifurcation. Second, they find the existence of a period doubling bifurcation. In this case, the solution can move from a stable to a periodic solution where periodicity doubles in successive bifurcations.

Next, Barnett and Duzhak (2014) explore whether their analysis of this simple setup carries over to the standard New Keynesian model. Then, the monetary policy rule is more complicated. The Taylor rule (see Taylor (1999)) has two components, which make the nominal interest rate a function of both inflation and the output gap. This extra component can lead the solution to become less prone to changes in its characteristics. Barnett and Duzhak (2014) use numerical methods to find that the standard New Keynesian model, with regime switching and a standard Taylor rule, does not exhibit any bifurcations for the range of feasible parameter combinations. While they do find a bifurcation boundary, it lies outside the relevant range of
parameter values with negative coefficients, while standard economics requires them to be positive.

Then, Barnett and Duzhak (2014) investigate whether a state-of-the-art hybrid Taylor rule exhibits any bifurcations. They solve the same baseline New Keynesian model, but use a Taylor rule that allows for forward looking response to inflation. After going through the same solution steps as in the previous case, they find that this model might exhibit a period-doubling bifurcation. The ideas from the basic setup thus carry over to the more prominent model of monetary policy.

5.3. Dynamics with a Simple Monetary Policy Rule

A central banker is implementing a policy, by which he reacts to inflation by changing an interest rate according to:

\[ i_t = \alpha(s_t)\pi_t, \]

where \( i_t \) is the nominal interest rate, \( \alpha(s_t) \) a state-dependent coefficient which changes with the policy regime \( s_t \), and \( \pi_t \) denotes the rate of inflation.

Barnett and Duzhak (2014) assume that there are two possible realizations for the policy regime \( s_t \). The policy regime determines the reaction to inflation, when setting the nominal interest rate. The linear reaction function to inflation evolves stochastically between two states \( s_t = 1 \) and \( s_t = 2 \), so that

\[
\alpha(s_t) = \begin{cases} 
\alpha_1 & \text{for } s_t = 1 \\
\alpha_2 & \text{for } s_t = 2,
\end{cases}
\]

where \( \alpha_i \) denotes different parameters that govern the aggressiveness of policy to combat inflation. As usual, an active policy regime is the one with policy parameter \( \alpha_i > 1 \).

The policy regime evolves according to a Markov chain where the transitional probabilities are given by the transition matrix with entries \( p_{ij} = P[s_t = j | s_{t-1} = i] \) where \( i, j = 1, 2 \). Following Davig and Leeper (2006), Barnett and Duzhak (2014) study the dynamics of this simple monetary policy rule by using the Fisher equation

\[ i_t = E_t\pi_{t+1} + r_t, \]

where \( r_t \) is the real interest rate.

The Fisher equation links the nominal interest rate to expected inflation and the real interest rate. Barnett and Duzhak (2014) use this relationship to solve for expected inflation which evolves as a function of the nominal and real interest rates. Plugging in for the policy rule
where the nominal interest rate is a function of inflation, they end up with the following dynamic system

\[
\begin{bmatrix}
E_t[\pi_{1t+1}] \\
E_t[\pi_{2t+1}]
\end{bmatrix} =
\begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
\alpha_1 & 0 \\
0 & \alpha_2
\end{bmatrix}
\begin{bmatrix}
\pi_{1t} \\
\pi_{2t}
\end{bmatrix} -
\begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
r_t
\end{bmatrix}.
\]

In this system, Barnett and Duzhak (2014) take the real interest rate as exogenously given. A fully specified macroeconomic model endogenizes this rate.

Barnett and Duzhak (2014) analyze this system of linear difference equations. Therefore, they replace the matrix multiplying the vector of inflation by explicitly computing the inverse of the transition matrix

\[
\begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
\alpha_1 & 0 \\
0 & \alpha_2
\end{bmatrix} = 
\begin{bmatrix}
\frac{p_{22}\alpha_1}{\Delta} & -\frac{p_{12}\alpha_2}{\Delta} \\
-\frac{p_{21}\alpha_1}{\Delta} & -\frac{p_{11}\alpha_2}{\Delta}
\end{bmatrix}
\]

where \(\Delta\) denotes the determinant of the transition matrix.

As is standard in the (bifurcation) analysis of difference equations, they study the economy with parameter certainty. Parameter certainty means that agents have no uncertainty about the level of inflation if a certain state occurs. This does not mean that agents know the level of inflation in the following period: the state of the policy regime determines inflation and the state of the policy regime itself switches with given probabilities. Using parameter certainty, Barnett and Duzhak (2014) replace the expected level of inflation conditional on a state occurring by the level of inflation.

Putting parameter certainty and the adjustments to the matrix and its determinant into our equation, Barnett and Duzhak (2014) can restate the system of linear difference equations as

\[
\begin{bmatrix}
\pi_{1t+1} \\
\pi_{2t+1}
\end{bmatrix} = 
\begin{bmatrix}
p_{22}\alpha_1 \\
p_{11}\alpha_2
\end{bmatrix}
\begin{bmatrix}
\frac{p_{11}p_{22} - p_{12}p_{21}}{p_{11}p_{22} - p_{12}p_{21}} \\
\frac{p_{11}p_{22} - p_{12}p_{21}}{p_{11}p_{22} - p_{12}p_{21}}
\end{bmatrix}
\begin{bmatrix}
\pi_{1t} \\
\pi_{2t}
\end{bmatrix} -
\begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
r_t
\end{bmatrix}.
\]

Since the entries in the transition matrix are probabilities, it follows that \(p_{11} + p_{21} = 1\) and \(p_{22} + p_{12} = 1\). Hence, \(\Delta = p_{11}p_{22} - p_{12}p_{21}\) as \(\Delta = p_{11} + p_{22} - 1\).

The key component of the dynamical system is the coefficient on current inflation. This Jacobian of the linear difference equation captures the evolution of expected inflation and thus the link between real and nominal interest rates. Given the calculations for the matrix on
inflation and the determinant of the transition matrix above, the Jacobian of the linear
difference equation is given by

\[
\mathbf{J} = \begin{bmatrix}
\frac{p_{22} \alpha_1}{p_{11} + p_{22} - 1} & -\frac{p_{12} \alpha_2}{p_{11} + p_{22} - 1} \\
-\frac{p_{21} \alpha_1}{p_{11} + p_{22} - 1} & \frac{p_{11} \alpha_2}{p_{11} + p_{22} - 1}
\end{bmatrix}
\]

To analyze the stability of the evolution of inflation and its dynamic properties, Barnett
and Duzhak (2014) compute the eigenvalues for the Jacobian matrix. Therefore, they compute
the characteristic polynomial \( P(\lambda) \) which is quadratic in this case given by

\[
P(\lambda) = \lambda^2 - b\lambda + c,
\]

where the coefficients are

\[
b = \frac{p_{22} \alpha_1 + p_{11} \alpha_2}{p_{11} + p_{22} - 1} \quad \text{and} \quad c = \frac{\alpha_1 \alpha_2}{p_{11} + p_{22} - 1}.
\]

The nature of the solution to quadratic equations is mainly determined by the
discriminant of the square root that appears in the formula. For the dynamics of inflation, the
determinant \( D \) is given by

\[
D = \left[ \frac{p_{22} \alpha_1 + p_{11} \alpha_2}{p_{11} + p_{22} - 1} \right]^2 - \frac{4 \alpha_1 \alpha_2}{p_{11} + p_{22} - 1}.
\]

Barnett and Duzhak (2014) are interested in the quality of the dynamics and whether
bifurcation exists, i.e. whether the quality of the solution can change drastically despite a
negligible change in the parameter values. For a Hopf bifurcation to exist, the discriminant \( D \)
must be negative, giving a rise to complex roots of the characteristic polynomial. To check
whether this is true, Barnett and Duzhak (2014) needed to solve for \( D < 0 \). Given that
\((p_{11} + p_{22} - 1)^2\) is always nonnegative, they can simply the inequality to \((p_{22} \alpha_1 + p_{11} \alpha_2)^2 - (p_{11} + p_{22} - 1)4 \alpha_1 \alpha_2 < 0\). The term on the left-hand side stays positive within the feasible set
of parameters. Therefore, a Hopf bifurcation which arises only when the roots are complex, is
not possible for this economy.

However, Barnett and Duzhak (2014) can check the possibility of a period doubling
bifurcation. This type of bifurcation occurs when the root equals -1 and it leads to the doubling
of the periodicity of the dynamic solution. The following lemma provides conditions for the
existence of the period doubling bifurcation (see Kuznetsov (1998), p.415).

**Lemma 5.1 (Period Doubling Bifurcation)** Suppose that a one dimensional system
\[ x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^1, \alpha \in \mathbb{R}^2, \]

with \( f \) smooth, so that at \( \alpha = 0 \) the fixed point \( x = 0 \), and let the period doubling bifurcation condition hold:

\[ \mu = f_x(0,0) = -1, c = 0. \]

Assume, that the following genericity conditions are satisfied:

**PD. 1** \[ D(0) = \left( \frac{1}{5} f_{x^5} + \frac{1}{2} f_x f_{x^4} - [f_x]^4 \right) (0,0) = 0; \]

**PD. 2** The map \( \alpha \mapsto (\mu(\alpha) + 1, c(\alpha))^T \) is regular at \( \alpha = 0 \), where \( c(\alpha) \) is given by

\[ c(\alpha) = b(\alpha) + \frac{2a^2(\alpha)}{\mu^2(\alpha) - \mu(\alpha)}. \]

Then there are smooth invertible coordinate and parameter changes transforming the system into

\[ \eta \mapsto -(1 + \beta_1) \eta + \beta_2 \eta^3 + s \eta^5 + O(\eta^6), \quad \text{where} \quad s = \text{sign}[D(0)] = \pm 1. \]

This system without \( O(\eta^6) \) terms is called the truncated normal form for the period doubling bifurcation.

For Barnett and Duzhak’s (2014) model, both conditions for the period doubling bifurcation hold. To find the combination of parameters that make the variable \( \mu \) from Lemma 5.1 equal to -1, they analyze the eigenvalues of the characteristic polynomial. The characteristic polynomial \( P(\lambda) \) has the following roots:

\[ \lambda_{1,2} = \frac{1}{2} \left[ \frac{\alpha_1 p_{22} + \alpha_2 p_{11}}{p_{11} + p_{22} - 1} \pm \sqrt{D} \right] \]

where \( D \) is the discriminant defined above.

If one of these roots is in the negative part of the unit circle, then there is a possibility of a period doubling bifurcation, given that the nondegeneracy conditions are satisfied.

From the equation for the roots above, Barnett and Duzhak (2014) get \( \mu = -1 \) whenever one of the roots equals -1. As a result, they can rearrange the expression to produce the condition

\[ \sqrt{(p_{22} \alpha_1 + p_{11} \alpha_2)^2 - (p_{11} + p_{22} - 1)^4 \alpha_1 \alpha_2} = 2(p_{11} + p_{22} - 1) + (p_{22} \alpha_1 + p_{11} \alpha_2) \]

that needs to hold for a period doubling bifurcation to occur.
Simplifying this expression gives us

\[ p_{11}(1 + \alpha_2) + p_{22}(1 + \alpha_1) + \alpha_1 \alpha_2 = 1. \]

This equation can be described as a bifurcation boundary. The bifurcation boundary is a function of the parameters of the dynamical model. Barnett and Duzhak (2014) chose critical bifurcation parameter to be \( p_{22}^c \).

To calibrate the economy, they use the values in Table 5.1. One of the policy regimes, regime 1, is active with a coefficient greater than 1 whereas regime 2 is a passive regime. The time preference factor \( \beta \), reaction of inflation to the output gap \( \kappa \), and the degree of relative risk aversion \( \sigma \) will only become relevant in the latter part of Barnett and Duzhak (2014).

They furthermore assume that the probability of staying in the active regime conditional on being in the active regime \( p_{11} = 0 \) is zero. Whenever regime 1 occurs, the economy will be sent to a passive regime with certainty.

<table>
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<th>Value</th>
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<td>( \alpha_2 )</td>
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<td>( \sigma )</td>
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</tbody>
</table>

Using these assumptions, Barnett and Duzhak (2014) determine the critical value for the transitional probability \( p_{22} \) being \( p_{22}^c \) = 0.1. Barnett and Duzhak (2014) use this point as a benchmark to trace out the bifurcation boundary. To obtain the entire bifurcation boundary, Barnett and Duzhak (2014) vary the other parameters, i.e. policy parameters \( \alpha_1 \) and \( \alpha_2 \), along with the probability of staying in the passive regime \( p_{22} \).
Consequently, Barnett and Duzhak (2014) demonstrate a period doubling bifurcation boundary as a function of the three control parameters $p_{22}$, $\alpha_1$ and $\alpha_2$. If $p_{22} = 1$, then the policy regime would be passive and stay passive indefinitely. In this case, no bifurcation can arise and the bifurcation boundary converges to zero. Second, for the case of $p_{22} = 0$, the two policy regime are identical in the likelihood with which they occur. The bifurcation boundary is thus symmetric along policy parameters $\alpha_1$ and $\alpha_2$.

Interestingly, however, a bifurcation boundary exists for all probabilities between these two extreme cases. In particular, if the policy reaction coefficient $\alpha_2$ of the passive regime is small, the policy response coefficient needs to be very large for a bifurcation to arise. For a very aggressive policy in the active regime, the rate of inflation will start to evolve in cycles despite the simple nature of its equation of motion.

5.4. New Keynesian Model with Regime Switching

This section describes the well-known equations for the standard New Keynesian setup as laid out in e.g. Woodford (2003) or Walsh (2003). The standard New Keynesian model traditionally consists of the forward-looking IS equation that describes the demand side of the economy

$$D_t = E_t D_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) + u_t^D$$

and the Phillips curve which represents the supply side

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t^S.$$

The IS curve relates the output gap $x_t$ to the nominal interest rate $i_t$ and expectations about the future output gap as well as inflation. The coefficient on the difference between the nominal interest rate and expected inflation, i.e. the coefficient on the real interest rate by the Fisher equation, is given by $\frac{1}{\sigma}$. This coefficient is the inverse of relative risk aversion which equals the elasticity of intertemporal substitution since preferences with constant relative risk aversion are assumed to derive the equations. The New Keynesian Phillips curve describes how inflation is driven by the output gap and expected inflation. Both equations for demand and supply side allow for a shock $u_t$.

The remaining equation to close the economy is a rule for monetary policy which takes the form described in Taylor (1999). According to this Taylor rule, the monetary authority sets the nominal interest rate by targeting both inflation and the output gap

$$i_t = \alpha(s_t) \pi_t + \gamma(s_t) x_t$$

where $\alpha_i$ governs the central bank’s reaction to inflation and $\gamma_i$ the reaction to the output gap.
After plugging the Taylor rule into the IS equation, the model can be written in matrix notation

\[ AY_{t+1} = BY_t + u_t \]

where the \( Y \) denotes the vector of variables \( Y = [\pi_{1t} \pi_{2t} x_{1t} x_{2t}]^T \) and \( u_t \) the vector of aggregate demand and supply shocks. The matrix multiplying next period’s endogenous variables, inflation and output in either policy regime, is given by

\[
A = \begin{bmatrix}
\beta p_{11} & \beta (1 - p_{22}) & 0 & 0 \\
\beta (1 - p_{11}) & \beta p_{22} & 0 & 0 \\
p_{11} & 1 - p_{22} & p_{11} & 1 - p_{22} \\
\frac{1 - p_{11}}{\sigma} & \frac{p_{22}}{\sigma} & 1 - p_{11} & p_{22}
\end{bmatrix}
\]

The matrix multiplying current period’s variables is given by

\[
B = \begin{bmatrix}
1 & 0 & -\kappa & 0 \\
0 & 1 & 0 & -\kappa \\
\frac{\alpha_1}{\sigma} & 0 & 1 + \frac{\gamma_1}{\sigma} & 0 \\
0 & \frac{\alpha_2}{\sigma} & 0 & 1 + \frac{\gamma_2}{\sigma}
\end{bmatrix}
\]

Pre-multiplying both sides of the equation by the inverse of the matrix \( A \), Barnett and Duzhak (2014) obtain the normal form

\[ Y_{t+1} = CY_t + A^{-1}u_t \]

used in the previous section, where \( C = A^{-1}B \).

Matrix \( C \) is the key element when performing bifurcation analysis of the model with a generalized Taylor rule.

Given the standard calibration provided in Table 5.1, Barnett and Duzhak (2014) search the parameter space of the elasticity of inflation with respect to the output gap \( \kappa \) and the Taylor coefficient \( \gamma_2 \) on the output gap in state 2 for possible bifurcations. In order to locate bifurcation values, Barnett and Duzhak (2014) needed to choose a free parameter. Parameters that describe the probabilities of regime change are held constant, while structural and policy parameters are varied.

At this stage, however, Barnett and Duzhak (2014) need to deviate from the path they took for the basic model of the previous section. While they deal with a two-by-two matrix in
the simple setup, they now have a four-dimensional dynamical system. As a result, the computation of the characteristic polynomial and its solution becomes more involved.

Barnett and Duzhak (2014) employ the software continuation package CONTENT developed by Yuri Kuznetsov and V.V. Levitin to trace out bifurcation boundaries for large dynamical systems. When using the software, they can show that neither a Hopf nor a periodic doubling bifurcation, nor the combination of them can occur for any feasible set of parameters. Barnett and Duzhak (2014) do find the bifurcation boundary. A period doubling-Hopf bifurcation occurs for parameter values $\gamma_2 = 0.179$ and $\kappa = -0.46$. However, since negative values for $\kappa$ are economically infeasible, this is not a relevant case. After tracing out the entire bifurcation boundary, it never crosses into the subspace of feasible parameter combinations. Hence they conclude that given the standard parameterization, the general Taylor rule leads to a structurally stable model.

5.5. New Keynesian Model with a Hybrid Monetary Policy Rule

This section provides an analysis of the state-of-art model of a monetary policy which consists of a hybrid rule which includes both a current-looking and a forward-looking component. Generally this type of rules can include the features of backward-looking rules such as past values of inflation or output gap, but Barnett and Duzhak (2014) limited their analysis to the following specification $i_t = \alpha(s_t)\pi_{t+1} + \gamma(s_t)x_t$.

This form of policy rule was proposed by Clarida, Gali and Gertler (1999) where they provide support for a superior to the policy implemented by the Federal Reserve. According to this rule, a policy maker is forward-looking with respect to inflation and current looking with respect to the output gap. Using this type of monetary policy rule in a New Keynesian setup produces the system linear difference equations $Y_{t+1} = DY_t$, where matrix $D$ is given by

$$D = \begin{bmatrix}
\frac{p_{22}}{\beta(-1+p_{22}+p_{11})} & \frac{-1+p_{22}}{\beta(-1+p_{22}+p_{11})} & \frac{-p_{22}k}{\beta(-1+p_{22}+p_{11})} & \frac{-(1+p_{22})k}{\beta(-1+p_{22}+p_{11})} \\
\frac{\beta(-1+p_{22}+p_{11})}{\beta(-1+p_{22}+p_{11})} & \frac{\beta(-1+p_{22}+p_{11})}{\beta(-1+p_{22}+p_{11})} & \frac{\beta(-1+p_{22}+p_{11})}{\beta(-1+p_{22}+p_{11})} & \frac{\beta(-1+p_{22}+p_{11})}{\beta(-1+p_{22}+p_{11})} \\
\frac{p_{22}(-1+\alpha_1)}{\alpha\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{22})(-1+\alpha_1)}{\alpha\beta(-1+p_{22}+p_{11})} & \frac{(-1+\alpha_2)p_{11}}{\alpha\beta(-1+p_{22}+p_{11})} & \frac{(-1+p_{11})(-1+\alpha_1)}{\alpha\beta(-1+p_{22}+p_{11})} \\
\frac{\alpha\beta(-1+p_{22}+p_{11})}{\alpha\beta(-1+p_{22}+p_{11})} & \frac{\alpha\beta(-1+p_{22}+p_{11})}{\alpha\beta(-1+p_{22}+p_{11})} & \frac{\alpha\beta(-1+p_{22}+p_{11})}{\alpha\beta(-1+p_{22}+p_{11})} & \frac{\alpha\beta(-1+p_{22}+p_{11})}{\alpha\beta(-1+p_{22}+p_{11})}
\end{bmatrix}$$

Barnett and Duzhak (2014) analyzed the coefficient matrix $D$ for possibilities of Hopf and period doubling bifurcations using the same steps as in the preceding section. Numerical analysis of this dynamic system leads to two findings. First, there is no possibility of a Hopf bifurcation.
Second, a period doubling bifurcation emerges. The findings are thus the same as for the simple economy.

To find a bifurcation boundary, a parameter needs to be varied. Barnett and Duzhak (2014) first vary parameter $\alpha_2$ while holding all other parameters constant. Assuming the standard calibration in Table 5.1, the critical value of parameter $\alpha_2$ is 0.00125. They use this point to trace out the bifurcation boundary. After tracking the first period doubling bifurcation point, Barnett and Duzhak (2014) chose the second parameter that is varied simultaneously with parameter $\alpha_2$. For that case, they can demonstrate a period doubling bifurcation boundary as a function of two control parameters, $\alpha_2$ and the risk aversion parameter $\sigma$. A period doubling bifurcation will occur for a very narrow set of parameters $\alpha_2$ corresponding to a passive reaction to future inflation, in the close proximity of zero. Similarly, they find a period doubling point for parameter $\kappa = 3.725$. After choosing a second parameter, $\sigma$, to be varied, Barnett and Duzhak (2014) compute the period doubling bifurcation boundary. Parameter $\kappa$ is a nonlinear function of the discount factor and the parameter responsible for the degree of price rigidity. It shows that the period doubling bifurcation will occur when the economy is characterized by a high level of price stickiness.

After analyzing further parameter combinations, Barnett and Duzhak (2014) find that a period doubling bifurcation is also possible for lower values of $\kappa$ accompanied by very high values of the policy parameter $\alpha_1$. In other words, aggressive reaction of the central bank to future inflation will lead to a period doubling bifurcation.

6. Open-Economy New Keynesian Models

6.1. Background


6.2. Introduction

With those two models, Barnett and Eryilmaz (2013, 2014) find that under a monetary policy rule, the degree of openness in New Keynesian models changes the location of bifurcation boundaries. In addition, the open economy framework brings about more complex
dynamics, along with a wider variety of qualitative behaviors and policy responses. As a result, the stratification of the confidence region, as previously seen in closed-economy New Keynesian models, remains an important research topic and policy risk to be considered in the context of the open-economy New Keynesian functional structures. Policy design needs to take into consideration that a change in monetary policy can produce bifurcation, that would be unanticipated without adequate prior econometric research.

As surveyed in section 6.3, Barnett and Eryilmaz (2014) ran bifurcation analyses on the Gali and Monacelli (2005) and have shown that in a broad class of open-economy New Keynesian models, the degree of openness has a significant role in equilibrium determinacy and emergence of bifurcations. Barnett and Eryilmaz (2014) acquired that result with various forms and timings of monetary policy rules. They established the conditions for Hopf bifurcation with each model, based on the Hopf Bifurcation Theorem. Numerical analyses are performed using our theoretical results and also to search for other types of bifurcation. Limit cycles and periodic behaviors are found, but in some cases only for unrealistic parameter values. The existence of the period doubling bifurcations are also identified by numerical analyses with CL MatCont.

Comparing the results from Barnett and Duzhak’s (2010) closed economy analysis does not provide clear conclusions about whether openness makes the New Keynesian model more sensitive to bifurcations. One reason is the Gali and Monacelli model’s broad set of parameters, including deep parameters relevant to the open economy. The fact that the studies use different sets of benchmark values for the parameters makes direct comparison harder. While the bifurcation phenomena exist in both open and closed economy New Keynesian models, Barnett and Eryilmaz (2014) do not find evidence that open economies are more vulnerable to the problem than closed economies.

As surveyed in section 6.4, Barnett and Eryilmaz (2013) showed that by varying the parameter $\phi_x$, while keeping the other parameters constant, the model of Clarida, Gali and Gertler (2002) is vulnerable to Hopf bifurcation at $\phi_x^*$. Barnett and Eryilmaz (2013) also show that the structural parameters, $w$, $v$, and $\delta$, play a significant role in determining the critical value of the bifurcation parameter, $\phi_x$. Their theoretical results need to be confirmed by subsequent numerical analysis to locate the Hopf bifurcation boundary and map its shape. But that numerical analysis is beyond the scope limited to determining the relevant theory.

A primary objective of the subsequent numerical analysis should be to determine whether the Hopf bifurcation boundary crosses relevant confidence regions of the model’s parameters. If so, a serious robustness problem would exist in dynamical inferences using the model. But even if the bifurcation boundary does not cross the confidence region, policy can move the location of the bifurcation boundary by changing the values of policy parameters.
Within this model, the central bank should react cautiously to changes in the rate of domestic inflation and the output gap. The central bank should particularly take into consideration the following structural parameters: price rigidity, \( \theta \), wage inflation, \( w \), and the wealth effect, \( v\sigma \), to avoid inducing instability from a possible Hopf bifurcation.

### 6.3. Gali and Monacelli Model

Gali and Monacelli (2005) define a small open economy to be “one among a continuum of infinitesimally small economies making up the world economy”. Thus, domestic policy does not affect the other countries and the world economy. In their model, each economy is assumed to have identical preferences, technology, and market structure, both consumers and firms are assumed to behave optimally. Consumers maximize expected present value of utility, while firms maximize profits.

The utility maximization problem yields the following dynamical intertemporal IS curve, which is a log-linear approximation to the Euler equation:

\[
D_t = E_t D_{t+1} - \left(1 + \frac{\omega - 1}{\sigma}\right) \left(r_t - E_t \pi_{t+1} - \bar{r}_t\right),
\]

where \( D_t \) is the gap between actual output and flexible-price equilibrium output, \( \bar{r}_t \) is the small open economy’s natural rate of interest, and \( \sigma\alpha = \sigma(1 - \alpha + \alpha\omega)^{-1} \) and \( \omega = \sigma\gamma + (1 - \alpha)(\sigma\eta - 1) \) are composite parameters. The lowercase letters denote the logs of the respective variables, \( \rho = \beta^{-1} - 1 \) denotes the time discount rate and \( a_t \) is the log of labor’s average product.

The maximization problem of the representative firm yields the aggregate supply curve, often called the New Keynesian Philips curve in log-linearized form:

\[
\pi_t = \beta E_t \pi_{t+1} + \frac{(1 - \beta\theta)(1 - \theta)}{\theta} \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi\right) x_t,
\]

Some coefficients of the open economy model depend on the parameters that are exclusive to the open economy framework, such as the degree of openness, terms of trade, and substitutability among domestic and foreign goods. The natural levels of output and interest rate depend upon both domestic and foreign disturbances, in addition to openness and terms of trade.

The model is closed by adding a simple (non-optimized) monetary policy rule, conducted by the monetary authority, such as:

\[
r_t = \bar{r}_t + \phi\pi_t + \phi_x x_t,
\]
where the coefficient $\phi_x > 0$ and $\phi_x > 0$ measures the sensitivity of the nominal interest rate to changes in output gap and inflation rate, respectively. In this form, the policy rule (6.3) is called the Taylor rule (Taylor (1993)). Various versions of the Taylor rule are often employed to design monetary policy in empirical DSGE models. Equations (6.1) and (6.2), in combination with a monetary policy rule such as equation (6.3), constitute a small open economy mode in the New Keynesian tradition.

To determine whether a Hopf bifurcation exists in the Gali and Monacelli model, the methodology is the same as in section 4 above; i.e. as in Barnett and Duzhak (2008,2010). Barnett and Eryilmaz (2013) first evaluate the Jacobian of the system at the equilibrium point, $\pi_t = x_t = 0$, for all $t = 1, 2, \ldots$, and then check whether the conditions of the Hopf Bifurcation Theorem are satisfied. For two-dimensional dynamic systems, the theorem is Theorem 1.1; for three-dimensional dynamic systems, the theorem is Theorem 4.1.

In computations Barnett and Eryilmaz (2013) always use CL MatCont for Hopf and all other forms of bifurcation that the program can detect. They consider contemporaneous, forward, and backward looking policy rules, as well as their hybrid combinations. They use the calibration values of the parameters as given in Gali and Monacelli (2005), which are $\beta = 0.99, \alpha = 0.4, \sigma = \omega = 1, \varphi = 3, \mu = 0.086$; and for the $N = 3$ policy parameters, they use $\phi_x = 0.125, \phi_\pi = 1.5, \text{ and } \phi_r = 0.5$.

i. Current-Looking Taylor Rule

Consider the following model, in which the first two equations describe the economy, while the third equation is the monetary policy rule followed by the central bank with $N = 2$ policy parameters:

$\pi_t = \beta E_t \pi_{t+1} + \mu \left(\frac{\sigma}{1 + \alpha(\omega - 1)} + \varphi\right) x_t, \quad (6.4)$

$x_t = E_t x_{t+1} - \frac{1 + \alpha(\omega - 1)}{\sigma} (r_t - E_t \pi_{t+1} - \bar{r}_t), \quad (6.5)$

$r_t = \bar{r}_t + \phi_\pi \pi_t + \phi_x x_t \quad (6.6)$

Rearranging the terms, the system can be written in the form $E_t y_{t+1} = Cy_t$.

$\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\mu}{\beta} + (1 + \alpha(\omega - 1)) \left(\frac{\beta \phi_x + \varphi \mu}{\beta \sigma}\right) & \frac{(\beta \phi_\pi - 1)(1 + \alpha(\omega - 1))}{\beta \sigma} \\ -\frac{\mu}{\beta} (\varphi + \frac{\sigma}{1 + \alpha(\omega - 1)}) & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}. \quad (6.7)$

Using Theorem 1.1, the conditions for the existence of Hopf bifurcation in the system (6.7) are presented in the following Proposition.
Proposition 6.1 Let $\Delta$ be the discriminant of the characteristic equation. Then system (6.7) undergoes a Hopf bifurcation, if and only if $\Delta < 0$ and

$$
\phi^*_x = \frac{\sigma(\beta - 1)}{1 + \alpha(\omega - 1)} - \mu \left( \phi + \frac{\sigma}{1 + \alpha(\omega - 1)} \right) \phi_x.
$$

(6.8)

In the closed economy case, the corresponding value of the bifurcation parameter is $\phi^*_x = \sigma(\beta - 1) - \kappa \phi_x$, as given by Barnett and Duzhak (2008). For $\alpha = 0$, proposition 6.1 gives the same result as the closed economy counterpart.

Barnett and Eryilmaz (2013) numerically find a period doubling bifurcation at $\phi_x = -2.43$ and a Hopf bifurcation at $\phi_x = -0.52$. Numerical computations indicate that the monetary policy rule equation (6.6) should have $\phi^*_x < 0$ for a Hopf or period doubling bifurcation to occur. That negative coefficient for the output gap in equation (6.6) would indicate a procyclical monetary policy: rising interest rates when the output gap is negative or vice versa. Schettkat and Sun (2009) identify situations, such as exchange rate stabilization or an underestimation of the potential output level, which can produce such a result; but otherwise it is difficult to rationalize a negative policy parameter on the output gap.

There is a large literature seeking to explain procyclicality in monetary policy. Demirel (2010), for example, shows that the existence of country spread can explain how optimal fiscal and monetary policies can be procyclical. Leith, Moldovan, and Rossi (2009) argue that with superficial habits, the optimal simple rule might exhibit a negative response to the output gap. Such a perverse policy response to output gap or to inflation can induce instability in the model. A countercyclical monetary policy, on the other hand, would be bifurcation-free and would yield more robust dynamical inferences with confidence regions not crossing a bifurcation boundary.

Under the current-looking Taylor rule, there is only one periodic solution, while the other solutions diverge from the periodic solution as $t \to \infty$. This periodic solution is called an unstable limit cycle. In conclusion, by assuming $\phi_x > 0$ and $\phi_\pi > 0$, the Gali and Monacelli Model with current-looking Taylor rule is not subject to bifurcation within the feasible parameter space, although bifurcation is possible within the more general functional structure of system (6.7).

ii. Current-Looking Taylor Rule With Interest Rate Smoothing

Consider the model consisting of equations (6.4) and (6.5), along with the following policy rule having $N = 3$ policy parameters:

$$
\eta_t = \eta_t^\ast + \phi_\pi \eta_t + \phi_x \eta_t^x + \phi_r \eta_{t-1}.
$$

(6.9)
Barnett and Eryilmaz (2013) write that system of three equations in the form
\[ E_t y_{t+1} = C y_t + \mathbf{d}_t \] as:
\[
\begin{bmatrix}
    E_{t+1} x_{t+1} \\
    E_{t+1} \pi_{t+1} \\
    E_{t+1} r_{t+1}
\end{bmatrix}
= C
\begin{bmatrix}
    x_t \\
    \pi_t \\
    r_t
\end{bmatrix}
+ \begin{bmatrix}
    -\frac{1-\alpha+\alpha \omega}{\sigma} \bar{r}_t \\
    0 \\
    E_t \bar{r}_{t+1} - \phi_x \frac{1-\alpha+\alpha \omega}{\sigma}
\end{bmatrix}
\] (6.10)

where \( C = \)
\[
\begin{bmatrix}
    \frac{\mu}{\beta} \left( 1 + \varphi \frac{1-\alpha+\alpha \omega}{\sigma} \right) + 1 & -\frac{1-\alpha+\alpha \omega}{\sigma \beta} & \frac{1-\alpha+\alpha \omega}{\sigma} \\
    -\frac{\mu}{\beta} \left( \frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) & \frac{1}{\beta} & 0 \\
    \phi_x + \frac{\mu}{\beta} \left( 1 + \varphi \frac{1-\alpha+\alpha \omega}{\sigma} \right) \left( \phi_x \frac{1-\alpha+\alpha \omega}{\sigma} - \phi_\pi \right) & -\frac{1}{\beta} \left( \phi_x \frac{1-\alpha+\alpha \omega}{\sigma} - \phi_\pi \right) & \phi_r + \phi_x \frac{1-\alpha+\alpha \omega}{\sigma}
\end{bmatrix}
\]

Assuming the system (6.10) has a pair of complex conjugate eigenvalues and a real-valued eigenvalue outside the unit circle, the following proposition states the conditions for the system to undergo a Hopf bifurcation.

**Proposition 6.2** The system (6.10) undergoes a Hopf bifurcation, if and only if the following transversality condition holds
\[ \frac{\partial \lambda_i(\phi)}{\partial \phi_j} \mid \phi_j = \phi_j^* \neq 0 \]

and also
\[
(i) \quad \phi_r - \beta < 0, \\
(ii) \quad \phi_r \left( \frac{\sigma (2+\mu+2 \beta)}{1-\alpha+\alpha \omega} + \varphi \mu \right) + \phi_x (\beta + 1) + \mu \left( \frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) (\phi_\pi + 1) + \\
\frac{2 \sigma}{1+\alpha(\omega-1)} < 0, \quad (6.12) \\
(iii) \quad \phi_r^2 \xi_4 + \phi_r \xi_3 + (\phi_x \phi_r + \phi_x) \xi_2 + \phi_\pi \xi_1 + \xi_0 = -1. \quad (6.13)
\]

Since condition (6.12) in Proposition 6.2 does not hold, Hopf bifurcation cannot occur in the Gali and Monacelli Model under the current-looking Taylor rule with interest rate smoothing. Barnett and Eryilmaz (2013) also analyze the system (6.10) for the existence of a period doubling bifurcation. Keeping the structural parameters and policy parameters, \( \phi_\pi \) and \( \phi_r \), constant at their baseline values, while varying the policy parameter \( \phi_x \) over a feasible range, they numerically find period doubling bifurcation at \( \phi_x = 0.83 \). When they consider \( \phi_\pi \)
as the bifurcation parameter, they numerically find a period doubling bifurcation at $\phi_\pi = 5.57$ and a branching point at $\phi_\pi = -1.5$. There is no bifurcation of any type at $(\omega, \alpha) = (0,1)$.

### iii. Forward-Looking Taylor Rule

Consider the model consisting of equations (6.4) and (6.5) along with the following policy rule:

$$ r_t = \tilde{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x E_t x_{t+1}. $$

Rearranging terms, the reduced system in normal form becomes

$$ E_t y_{t+1} = C y_t $$

with

$$ C = \begin{bmatrix}
\frac{\beta \sigma - \left(\mu \sigma + \mu \phi(1+\alpha(\omega-1))\right)(\phi_\pi-1)}{\beta \sigma - \beta \phi_x (1+\alpha(\omega-1))} & \frac{(\phi_\pi-1)(1+\alpha(\omega-1))}{\beta \sigma - \beta \phi_x (1+\alpha(\omega-1))} \\
\frac{\mu \sigma + \mu \phi (1+\alpha(\omega-1))}{\beta + \alpha \beta (\omega-1)} & \frac{1}{\beta}
\end{bmatrix}. $$

**Proposition 6.3** The system (6.15) undergoes a Hopf bifurcation, if and only if $\Delta < 0$ and

$$ \phi_x^* = \frac{\beta^{-1}}{\beta} \frac{\sigma}{1+\alpha(\omega-1)}. $$

Numerical analysis with CL MatCont indicates a period doubling bifurcation at $\phi_x = 1.913$ and a Hopf bifurcation at $\phi_x = -0.01$. The system has a periodic solution at $\phi_\pi = 2.8$ and $\phi_x = 0$. The origin is a stable spiral point. Any solution that starts around the origin in the phase plane will spiral toward the origin. The origin is a stable sink.

### iv. Pure Forward-Looking Inflation Targeting

Consider the model consisting of equations (6.4) and (6.5) along with the following policy rule:

$$ r_t = \tilde{r}_t + \phi_\pi E_t \pi_{t+1}. $$

Rearranging the terms, the following reduced system is acquired in normal form

$$ E_t y_{t+1} = C y_t $$

with

$$ C = \begin{bmatrix}
1 - \left(\frac{\mu}{\beta} + \frac{\phi \mu (1+\alpha(\omega-1))}{\beta \sigma}\right)(\phi_\pi - 1) & \frac{(\phi_\pi-1)(1+\alpha(\omega-1))}{\beta \sigma} \\
- \frac{\mu}{\beta} \left(\frac{\sigma}{1+\alpha(\omega-1)} + \phi\right) & \frac{1}{\beta}
\end{bmatrix}. $$
The solution path for $\beta = 1$ and $\phi_\pi = 8$ is periodic and oscillates around the origin without converging or diverging. The origin is a stable center.

**Proposition 6.4** The system (6.18) undergoes a Hopf bifurcation, if and only if

\[ \Delta < 0 \text{ and } \beta^* = 1. \quad (6.19) \]

This result shows that setting the discount factor equal to 1 puts the system on the Hopf bifurcation boundary and creates instability. Barnett and Eryilmaz (2013) also numerically find a period doubling bifurcation at $\beta = -0.91$. But that point is outside the feasible parameter space subset. Furthermore, Hopf bifurcation appears at $\beta = 1$ regardless of the values of $\alpha$ and $\omega$. Bifurcation analysis in an open economy framework yields the same results as in the closed economy case under forward-looking inflation targeting. In section 4, we have observed that Barnett and Duzhak (2010) report a Hopf bifurcation at $\beta = 1$ for the closed economy case. But setting the discount factor at 1 is not justifiable for a New Keynesian model, whether within an open or closed economy framework.

There is only one periodic solution, and other solutions diverge from the periodic solution at $t \to \infty$. This periodic solution is an unstable limit cycle. If the policy parameter $\phi_\pi$, is varied, while setting $\beta = 1$ and keeping the other parameters constant at their baseline values, Barnett and Eryilmaz (2013) numerically find a Hopf bifurcation at $\phi_\pi = 1.0176$, a period doubling bifurcation at $\phi_\pi = 12.76$, and a branching point at $\phi_\pi = 1$.

v. **Backward-Looking Taylor Rule**

Consider the model consisting of equations (6.4) and (6.5) along with the following policy rule:

\[ r_t = \bar{r}_t + \phi_\pi \pi_{t-1} + \phi_x x_{t-1}. \quad (6.20) \]

Rearranging the terms, Barnett and Eryilmaz (2013) acquired the following reduced system in normal form $E_t y_{t+1} = Cy_t + d_t$:

\[ E_t y_{t+1} = Cy_t + \begin{bmatrix} \frac{1+\alpha(\omega-1)}{\sigma} \bar{r}_t \\ 0 \\ E_t \bar{r}_{t+1} \end{bmatrix}, \quad (6.21) \]

where

\[
C = \begin{bmatrix}
\frac{\mu}{\beta} \left(1 + \frac{\phi(1+\alpha(\omega-1))}{\sigma}\right) + 1 & -\frac{1+\alpha(\omega-1)}{\beta \sigma} & \frac{1+\alpha(\omega-1)}{\sigma} \\
-\frac{\mu}{\beta} \left(\frac{\sigma}{1+\alpha(\omega-1)} + \phi\right) & \frac{1}{\beta} & 0 \\
\phi_x & \phi_\pi & 0
\end{bmatrix}.
\]
Proposition 6.5  The system (6.21) undergoes a Hopf bifurcation, if and only if the transversality condition holds $\frac{\partial\lambda_j(\Phi^*\Phi)}{\partial\phi_j} |_{\Phi=\Phi^*} \neq 0$ for some $j$, and the following conditions also are satisfied:

(i) $\phi_x + \phi_{\pi} \mu \left( \frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) - \frac{\beta\sigma}{1+\alpha(\omega-1)} < 0$, (6.22)

(ii) $\phi_x (\beta - 1) + \mu \left( \frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) (1 - \phi_{\pi}) < 0$, (6.23)

(iii) $\left( \phi_x + \phi_{\pi} \mu \left( \frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) \right)^2 + \left( \phi_x + \phi_{\pi} \mu \left( \frac{\sigma}{1+\alpha(\omega-1)} + \varphi \right) \right) \xi_1 + 2 \phi_x \xi_2 = \xi_3$. (6.24)

Barnett and Eryilmaz (2013) numerically detect a period doubling bifurcation at $\phi_x = 1.91$. Starting from the point $\phi_x = 1.91$, they construct the period doubling boundary by varying $\phi_x$ and $\phi_{\pi}$ simultaneously. Their numerical analysis with CL MatCont detects a codimension-2 fold-flip bifurcation (LPPD) at $(\phi_x, \phi_{\pi}) = (0.94, 2.01)$ and a flip-Hopf bifurcation (PDNS) at $(\phi_x, \phi_{\pi}) = (-6.98, 3.36)$. But treating the policy parameter $\phi_{\pi}$ as the potential source of bifurcation, while keeping the other parameters constant at their benchmark values, our numerical analysis with CL MatCont indicates a period doubling bifurcation at $\phi_{\pi} = 11.87$. Barnett and Eryilmaz (2013) find period doubling bifurcation at relatively large values of the parameter $\phi_{\pi}$, but still within the subset of the parameter space defined to be feasible by Bullard and Mitra (2002).

vi. Backward-Looking Taylor Rule with Interest Rate Smoothing

Consider the model consisting of equations (6.4) and (6.5) along with the following policy rule:

$$r_t = \tilde{r}_t + \phi_{\pi} r_{t-1} + \phi_x x_{t-1} + \phi_{\pi} r_{t-1}.$$ (6.25)

Rearranging the terms, we have the following reduced system in normal form $E_t y_{t+1} = C y_t + d_t$:

$$E_t y_{t+1} = C y_t + \begin{bmatrix} 0 & -\frac{1+\alpha(\omega-1)}{\sigma} \tilde{r}_t \\ 0 & E_t \tilde{r}_{t+1} \end{bmatrix},$$ (6.26)

where
\[
C = \begin{bmatrix}
\frac{\mu}{\beta} \left(1 + \frac{\varphi(1+\alpha(\omega-1))}{\sigma}\right) + 1 & -\frac{1+\alpha(\omega-1)}{\beta\sigma} & \frac{1+\alpha(\omega-1)}{\sigma} \\
-\mu \left(1 + \frac{\varphi(1+\alpha(\omega-1))}{\sigma}\right) & \frac{1}{\beta} & 0 \\
\phi_x & \phi_{\pi} & \phi_r \\
\end{bmatrix}.
\]

**Proposition 6.6** The system (6.26) undergoes a Hopf bifurcation, if and only if the transversality condition holds \(\frac{\partial [\lambda_i(\Phi)]}{\partial \phi_j}|_{\Phi=\Phi^*} \neq 0\) for some \(j\), and the following conditions also are satisfied:

\[(i) \left| \frac{\phi_x - \phi_{\pi} \frac{\sigma}{1+\alpha(\omega-1)} + \phi_{\pi} \left(\frac{\sigma\mu}{1+\alpha(\omega-1)} + \varphi\mu\right)}{\beta\sigma} \right| < 1,\]

with \(\phi_x - \phi_{\pi} \xi_2 + \phi_{\pi} \xi_3 < \frac{\beta\sigma}{1+\alpha(\omega-1)}\), and \(\phi_r < \phi_x \xi_2 + \phi_{\pi} \xi_1 + \beta\),

\[(ii) \left| \phi_x \frac{1 - \alpha + \alpha\omega}{\beta\sigma} - \phi_r \frac{1}{\beta} + \phi_{\pi} \mu \left(\frac{1}{\beta} + \varphi \frac{1 - \alpha + \alpha\omega}{\beta\sigma}\right) - \left(\phi_r + \frac{1 + \mu}{\beta} + \varphi \mu \frac{1 - \alpha + \alpha\omega}{\beta\sigma} + 1\right) \right| < 1 + \phi_r \left(\frac{1 + \mu}{\beta} + \varphi \mu \frac{1 - \alpha + \alpha\omega}{\beta\sigma} + 1\right) - \phi_x \frac{1 - \alpha + \alpha\omega}{\beta\sigma} + \frac{1}{\beta},\]

with \(\phi_x \xi_2 + \phi_{\pi} \xi_1 - (1 + \phi_r) \xi_0 < 0\), and \(\phi_x \xi_3 - \xi_4 (\phi_{\pi} + \phi_r - 1) < 0\),

\[(iii) \phi_r \left(\frac{1 + \mu}{\beta} + \varphi \mu \frac{1 - \alpha + \alpha\omega}{\beta\sigma} + 1\right) - \phi_x \frac{1 - \alpha + \alpha\omega}{\beta\sigma} + \frac{1}{\beta} + \left(\phi_x \frac{1 - \alpha + \alpha\omega}{\beta\sigma} - \phi_r \frac{1}{\beta} + \phi_{\pi} \mu \left(\frac{1}{\beta} + \varphi \frac{1 - \alpha + \alpha\omega}{\beta\sigma}\right)\right) \left(\phi_r + \frac{1 + \mu}{\beta} + \varphi \mu \frac{1 - \alpha + \alpha\omega}{\beta\sigma} + 1\right) = 1 - \left(\phi_x \frac{1 - \alpha + \alpha\omega}{\beta\sigma} - \phi_r \frac{1}{\beta} + \phi_{\pi} \mu \left(\frac{1}{\beta} + \varphi \frac{1 - \alpha + \alpha\omega}{\beta\sigma}\right)\right)^2.\]

Given the benchmark values of the parameters and setting \(\phi_r = 0.5\), a period doubling bifurcation is detected numerically at \(\phi_x = 3\). When \(\phi_r = 1\), period doubling bifurcation occurs at \(\phi_x = 4.09\).

Our numerical analysis with CL MatCont indicates codimension-2 fold-flip bifurcations at \((\phi_x, \phi_{\pi}) = (0.41, 3.19)\) and at \((\phi_x, \phi_{\pi}) = (0.78, -0.52)\), as well as flip-Hopf bifurcations at \((\phi_x, \phi_{\pi}) = (-10.44, 5.04)\) and \((\phi_x, \phi_{\pi}) = (-0.74, -1.23)\). Bifurcation disappears at \((\alpha, \omega) = (1, 0)\).

vii. **Hybrid Taylor Rule**

Consider the model consisting of equations (6.4) and (6.5) along with the following policy rule:
\[ r_t = \tilde{r}_t + \phi_\pi E_t \pi_{t+1} + \phi_x x_t \quad (6.27) \]

Rearranging terms, Barnett and Eryilmaz (2013) provide the reduced system in normal form,

\[ E_t y_{t+1} = Cy_t \quad (6.28) \]

with

\[
C = \begin{bmatrix}
\frac{\beta \phi_x + \mu (1 + \alpha (\omega - 1) + \varphi)(1 - \phi_\pi)}{1 + \alpha (\omega - 1)} + 1 & \frac{(\phi_\pi - 1)(1 + \alpha (\omega - 1))}{\beta \sigma} \\
\frac{\beta \sigma}{1 + \alpha (\omega - 1)} & \frac{\beta}{\sigma}
\end{bmatrix}
\]

**Proposition 6.7** The system (6.28) exhibits a Hopf bifurcation, if and only if \( \Delta < 0 \) and

\[
\phi_x^* = \frac{\sigma(\beta - 1)}{1 + \alpha (\omega - 1)} \quad (6.29)
\]

Numerical analysis with CL MatCont indicates a period doubling bifurcation at \( \phi_x = -1.92 \), as well as a Hopf bifurcation at \( \phi_x = -0.01 \), given the benchmark values of the system parameters. Under the hybrid Taylor rule, values of the bifurcation parameters are outside the feasible region of the parameter space, since the New Keynesian economic theory normally assumes positive values for policy parameters, therefore, Barnett and Eryilmaz (2013) conclude that the feasible set of parameter values for \( \phi_x \) does not include a bifurcation boundary.

### 6.4. Clarida, Gali and Gertler Model

Barnett and Eryilmaz (2013) investigate the possibility of bifurcation in the open-economy New Keynesian model developed by Clarida, Gali, and Gertler (2002). Clarida, Gali and Gertler (2002) developed a two-country version of a small open economy model, which is based on Clarida, Gali and Gertler (2001) and Gali and Monacelli (1999). Let \( x_t \) denote the output gap, \( \pi_t^h \) the inflation rate for domestically produced goods and services, and \( r_t \) the nominal interest rate, with \( E_t \) being the expectation operator and \( \tilde{r}_t \) denoting the small open economy’s natural rate of interest. The lowercase letter denotes the logs of the respective variables. Then, following Walsh (2003, pp.539-540), the model of Clarida, Gali, and Gertler (2002) can be written in the reduced form as follows:

\[
\pi_t^h = \beta E_t \pi_{t+1}^h + \delta \left[ \sigma + \eta + \frac{\varphi}{1 + w} \right] x_t, \quad (6.30)
\]

\[
x_t = E_t x_{t+1} - \left( \frac{1 + w}{\sigma} \right) (r_t - E_t \pi_{t+1}^h - \tilde{r}_t), \quad (6.31)
\]
\[ r_t = \bar{r}_t + \phi_r \pi_t^h + \phi_x x_t. \]  

(6.32)

The coefficients \( \phi_x > 0 \) and \( \phi_r > 0 \) are the policy parameters, which measure the sensitivity of the nominal interest rate to changes in output gap and inflation rate, respectively. In addition, \( \delta = [(1 - \theta)(1 - \beta \theta)]/\theta \) is a composite parameter with \( \theta \) representing the probability that a firm holds its price unchanged in a given period of time, while \( 1 - \theta \) is the probability that a firm resets its price. The parameter \( \eta \) denotes the wage elasticity of labor demand, and \( \sigma^{-1} \) denotes the elasticity of intertemporal substitution. The parameter \( w \) denotes the growth rate of nominal wages, \( \rho = \beta^{-1} - 1 \) is the time discount rate, and \( v \) is the population size in the foreign country, with \( 1 - v \) being the population size of the home country. Wealth effect is captured by the term \( v \sigma \).

Equation (6.30) is an inflation adjustment equation for the aggregate price of domestically produced goods. Equation (6.31) is the dynamic IS curve, which is derived from the Euler condition of the consumers' optimization problem. The monetary policy rule (6.32) is a domestic-inflation-based current-looking Taylor rule, which completes the model.

Substituting (6.32) for \( r_t - \bar{r}_t \) into the equation (6.31), Barnett and Eryilmaz (2013) reduce the system to a first order dynamic system in two equations for domestic inflation and output gap, given by:

\[
\pi_t^h = \beta E_t \pi_{t+1}^h + \delta \left[ \sigma + \eta + \left( \frac{v \sigma}{1+w} \right) \right] x_t,
\]

\[
x_t = E_t x_{t+1} - \left( \frac{1+w}{\sigma} \right) \left( \phi_r \pi_t^h + \phi_x x_t - E_t \pi_{t+1}^h \right).
\]

Clearly, \( x_t = \pi_t^h = 0 \) for all \( t \) constitutes a solution (equilibrium) to the system. They can write the system in the standard form as

\[
A E_t y_{t+1} = B y_t.
\]

(6.33)

Then, premultiplying the terms on the right hand side by the inverse of the matrix \( A \), the system can be reduced to the form \( E_t y_{t+1} = Cy_t \), where \( C = A^{-1}B \), as follows:

\[
\begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1}^h \end{bmatrix} = C \begin{bmatrix} x_t \\ \pi_t^h \end{bmatrix}
\]

(6.34)

where

\[
C = \begin{bmatrix} 1 + \frac{(1+w)\phi_x}{\sigma} + \delta(1+w) \left( \sigma + \eta + \left( \frac{v \sigma}{1+w} \right) \right) \frac{1}{\beta \sigma} & \frac{(1+w)\phi_r}{\sigma} - \frac{(1+w)}{\beta \sigma} \\ -\delta(\sigma + \eta + \left( \frac{v \sigma}{1+w} \right) \frac{1}{\beta} & \frac{1}{\beta} \end{bmatrix}.
\]
Proposition 6.8 Let $\Delta$ be the discriminant of the characteristic equations. Then the system (6.34) undergoes a Hopf bifurcation, if and only if $\Delta < 0$ and

$$\phi^*_x = \frac{\beta \sigma - 1}{1 + w} - \phi_t \left( \frac{\delta \sigma (1 + \nu + w)}{1 + w} + \delta \eta \right).$$

(6.35)


Note that, the model of Clarida, Gali and Gertler (2002) differs in several aspects from the Galo and Monacelli (2005) model, which we used in another study. Additional parameters exist in the former model. In that model, the parameters $w, \nu$ and $\delta$ play an important role in determining the critical value of the bifurcation parameter, as Barnett and Eryilmaz (2013) have shown. The degree to which the two models differ depends upon the parameter settings. But it is clear that numerical implementation of the theory to locating Hopf bifurcation boundaries in the Clarida, Gali, and Gerler (2002) model would be a challenging project.

7. Two Endogenous Growth Models

7.1. Background


7.2. Introduction

Section 7.3 provides the Barnett and Ghosh (2013b) results for the Uzawa-Lucas endogenous growth model, while section 7.4 provides the Barnett and Ghosh (2013a) results for a variant of Jones (2002) semi-endogenous growth model.

The Uzawa-Lucas endogenous growth model (Uzawa (1965) and Lucas (1988)), upon which many others have been built, is among the most important endogenous growth models.
The model has two sectors: the human capital production sector and the physical capital production sector producing human capital and physical capital, respectively. Individuals have the same level of work qualification and expertise \( (H) \). They allocate some of their time to producing final goods and dedicate the remaining time to training and studying. The model is solved from a centralized social planner perspective as well as in the model’s decentralized market economy form.

Section 7.3 first provides the steady states, then the stability properties of the limit cycles generated by Hopf bifurcations and of other kinds of detected bifurcations. Barnett and Ghosh (2013b) use the numerical continuation package MatCont and locate Hopf and transcritical bifurcation boundaries using Mathematica. They show the existence of Hopf bifurcation, branch point bifurcation, limit point cycle bifurcation, and period doubling bifurcations. While these are all local bifurcations, the presence of global bifurcation is confirmed as well. They find evidence that the model could produce chaotic dynamics through the detected series of period-doubling bifurcations known to converge to chaos, but their analysis cannot definitively confirm that conjecture.

Section 7.4 surveys Barnett and Ghosh (2013a) incorporation of human capital accumulation into a Jones model. They explicitly takes into account the possibility that the investment in skill acquisition by agents might be positively, negatively, or not influenced at all by technological progress. Hence the direction of technological progress is ultimately driven by human capital investment (Bucci, 2008). Compared to Bucci (2008), Barnett and Ghosh (2013a) introduce the possibility of decreasing returns to scale associated with human capital and with time spent accumulating human capital in the production equation. The assumption of decreasing returns to scale is necessary to account for the scale effects in the model. Also, the introduction of such a human capital accumulation equation permits a closed form solution for the steady state of the modified model.

Barnett and Ghosh (2013a) used Matcont to analyze the bifurcation scenario. They showed the existence of Hopf, branch point, limit point of cycles, Bogdanov-Takens and generalized Hopf bifurcations within the feasible parameter range of the model. The choice of certain parameters in locating various bifurcation boundaries emphasizes the role played by human capital in such a model. The engine of growth is technological progress, which in turn, is driven by human capital investment. The presence of this interdependent relationship, by which the level of technological progress influences the rate of human capital accumulation, which in turn determines the growth rate of technology, creates the possibility of a multitude of dynamics in the model. Hence, the parameters in the human capital accumulation among other equations play a key role in determining the dynamics of the model. It could even lead
the economy to an unstable equilibrium, such that the balanced growth path may never be achieved.

In both models, it is important to recognize that bifurcation boundaries do not necessarily separate stable from unstable solution domains. Bifurcation boundaries can separate one kind of unstable dynamics domain from another kind of unstable dynamics domain, or one kind of stable dynamics domain from another kind (called soft bifurcation), such as bifurcation from monotonic stability to damped periodic stability or from damped periodic to damped multiperiodic stability. There are not only an infinite number of kinds of unstable dynamics, some very close to stability in appearance, but also an infinite number of kinds of stable dynamics. Hence subjective prior views on whether the economy is or is not stable provide little guidance without mathematical analysis of model dynamics.

7.3. Uzawa-Lucas Endogenous Growth Model

The production function in the physical sector is defined as follows:

\[ Y = AK^\alpha (\varepsilon hL)^{1-\alpha}h^\zeta_a, \quad 0 < \alpha < 1, \]

where \( Y \) is output, \( A \) is constant technology level, \( K \) is physical capital, \( \alpha \) is the share of physical capital, \( L \) is labor, and \( h \) is human capital per person. In addition, \( \varepsilon \) and \( 1 - \varepsilon \) are the fraction of labor time devoted to producing output and human capital, respectively, where \( 0 < \varepsilon < 1 \). Observe that \( \varepsilon hL \) is the quantity of labor, measured in efficiency units, employed to produce output, and \( h^\zeta_a \) measures the externality associated with average human capital of the work force, \( h_a \), where \( \zeta \) is the positive externality parameter in the production of human capital. In per capita terms, \( y = Ak^\alpha (\varepsilon h)^{1-\alpha}h^\zeta_a \).

The physical capital accumulation equation is

\[ \dot{K} = AK^\alpha (\varepsilon hL)^{1-\alpha}h^\zeta_a - C - \delta K. \]

In per capita terms,

\[ \dot{k} = Ak^\alpha (\varepsilon h)^{1-\alpha}h^\zeta_a - c - (n + \delta)k, \]

and the human capital accumulation equation is \( \dot{h} = \eta(1 - \varepsilon)h \), where \( \eta \) is defined as schooling productivity.

The decision problem is

\[ \max_{c_t, \varepsilon_t} \int_t^\infty e^{-(\rho - n)t} \left[ c(r)^{1-\alpha} - 1 \right] dt \quad (7.1) \]
subject to
\[ \dot{k} = Ak^\alpha (\varepsilon h)^{1-\alpha} h^\zeta - c - (n + \delta)k \quad (7.2) \]
and
\[ \dot{h} = \eta (1 - \varepsilon) h \quad (7.3) \]
where \( \rho \) \((\rho > n > 0)\) is the subjective discount rate, and \( \sigma \geq 0 \) is the reciprocal of the intertemporal elasticity of substitution in consumption.

In solving the maximization problem (7.1) subject to (7.2) and (7.3), the social planner takes into account the externality associated with human capital. From the first order conditions, Barnett and Ghosh (2013b Appendix 2) derive the equations describing the economy of the Uzawa-Lucas model from a social planner’s perspective

\[ \frac{\dot{k}}{k} = Ak^{\alpha - 1} \varepsilon^{1-\alpha} h^{1-\alpha + \zeta} - \frac{c}{k} - (n + \delta), \]
\[ \frac{\dot{h}}{h} = \eta (1 - \varepsilon), \]
\[ \frac{\dot{c}}{c} = \frac{\alpha Ak^{\alpha - 1} \varepsilon^{1-\alpha} h^{1-\alpha + \zeta} - (\rho + \delta)}{\sigma}, \]
\[ \frac{\dot{\varepsilon}}{\varepsilon} = \eta \frac{(1 - \alpha + \zeta)}{1 - \alpha} \varepsilon + \eta \frac{(1 - \alpha + \zeta)}{\alpha} - \frac{c}{k} + \frac{(1 - \alpha)}{\alpha} (n + \delta), \]
\[ \frac{\dot{L}}{L} = n. \]

Let \( m = \frac{Y}{K} = Ak^{\alpha - 1} \varepsilon^{1-\alpha} h^{1-\alpha + \zeta} \) \& \( g = \frac{c}{k} \). Taking logarithms of \( m \) and \( g \) and differentiating with respect to time, equation (4) and (5) define the dynamics of Uzawa-Lucas model

\[ \frac{\dot{m}}{m} = -(1 - \alpha) m + \frac{1-\alpha}{\alpha} (n + \delta) + \eta \frac{(1-\alpha+\zeta)}{\alpha}, \quad (7.4) \]
\[ \frac{\dot{g}}{g} = \left( \frac{\alpha}{\sigma} - 1 \right) m - \frac{\rho}{\sigma} - \delta \left( \frac{1}{\alpha} - 1 \right) + g + n. \quad (7.5) \]

The steady state \((m^*, g^*)\) is given by \( \dot{m} = \dot{g} = 0 \) and derived to be
\[ m^* = \eta \frac{(1 - \alpha + \zeta)}{\alpha} + \frac{(n + \delta)}{\alpha}. \]
\[
g^* = \frac{\rho - n}{\sigma} + \frac{1 - \alpha}{\alpha} (n + \delta) + \eta \frac{(1 - \alpha + \zeta)}{\alpha} (\sigma - \alpha) - \frac{\zeta}{\alpha} \varepsilon.
\]

A unique steady state exists, if
\[
\Lambda = \frac{(1 - \alpha + \zeta)}{\alpha} (\sigma - 1) \eta (1 - \varepsilon) + \rho > 0.
\]

In addition, this inequality condition for \( \Lambda \) provides the necessary and sufficient transversality condition for the consumer’s utility maximization problem (see Appendix 1 in Barnett and Ghosh (2013b)). Following the footsteps of Barro and Sala-i-Martín (2003) and Mattana (2004), it can be shown that social planner solution is saddle path stable. Linearizing around the steady state, \( s^* = (m^*, g^*) \), the local stability properties of the system defined by equations (7.4) and (7.5) can be found. The result is
\[
\begin{bmatrix}
\hat{m} \\
\hat{g}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial m}{\partial m} |_{s^*} & \frac{\partial m}{\partial g} |_{s^*} \\
\frac{\partial \hat{g}}{\partial m} |_{s^*} & \frac{\partial \hat{g}}{\partial g} |_{s^*}
\end{bmatrix} \begin{bmatrix}
m_t - m^* \\
g_t - g^*
\end{bmatrix},
\]
where
\[
J_s = \begin{bmatrix}
-(1 - \alpha) m^* & 0 \\
\left(\frac{\alpha}{\sigma} - 1\right) g^* & g^*
\end{bmatrix}.
\]

As \( m^* > 0 \) and \( g^* > 0 \), it follows that \( det(J_s) = -(1 - \alpha) m^* g^* < 0 \). Hence the saddle path is stable. From the first order conditions (see Appendix 3, Barnett and Ghosh (2013b)) with \( h = h_\alpha \), the equations describing the Uzawa-Lucas’s decentralized model can be found.

Let \( m = \frac{Y}{K} \) and \( g = \frac{C}{K} \). Taking logarithms of \( m \) and \( g \) and differentiating with respect to time, the following three equations define the dynamics of Uzawa Lucas model
\[
\begin{align*}
\frac{\dot{m}}{m} &= -(1 - \alpha) m + \frac{(1 - \alpha)}{\alpha} (n + \delta) + \eta \frac{(1 - \alpha + \zeta)}{\alpha} - \frac{\zeta}{\alpha} \varepsilon, \quad (7.6) \\
\frac{\dot{g}}{g} &= \left(\frac{\alpha}{\sigma} - 1\right) m - \frac{\rho}{\sigma} - \delta \left(\frac{1}{\sigma} - 1\right) + g + n, \quad (7.7) \\
\frac{\dot{\varepsilon}}{\varepsilon} &= \eta \frac{(\alpha - \zeta)}{\alpha} \varepsilon + \eta \frac{(1 - \alpha + \zeta)}{\alpha} - g + \frac{(1 - \alpha)}{\alpha} (n + \delta). \quad (7.8)
\end{align*}
\]

The steady state \( (m^*, g^*, \varepsilon^*) \) is given by \( \dot{m} = \dot{g} = \dot{\varepsilon} = 0 \) and derived to be
\[ \varepsilon^* = 1 - \frac{(1 - \alpha)(\rho - n - \eta)}{\eta[\zeta - \sigma(1 - \alpha + \zeta)]} \]

\[ m^* = \eta \frac{[1 - \alpha + \zeta(1 - \varepsilon^*)]}{\alpha(1 - \alpha)} + \frac{n}{\alpha} \]

\[ g^* = \eta \frac{[1 - \alpha + \zeta(1 - \varepsilon^*) + \alpha \varepsilon^*]}{\alpha(1 - \alpha)} + \frac{n(1 - \alpha)}{\alpha} \]

Note that as shown by Benhabib and Perli (1994), the inequality, \( 0 < \frac{(1 - \alpha)(\rho - n - \eta)}{\eta[\zeta - \sigma(1 - \alpha + \zeta)]} < 1 \), is necessary for \( 0 < \varepsilon^* < 1 \). A unique steady state exists, if

\[ \Lambda = \frac{(1 - \alpha + \zeta)}{\alpha}(\sigma - 1)\eta(1 - \varepsilon) + \rho > 0, \]

for \( 0 < \varepsilon < 1 \). In addition, \( \Lambda \) is the necessary and sufficient for the transversality condition to hold for the consumer's utility maximization problem (appendix 1 in Barnett and Ghosh (2013b)), and \( 0 < \varepsilon^* < 1 \) is necessary for \( m^*, g^* > 0 \). Linearizing the system ((7.6), (7.7), (7.8)) around the steady state, \( s^* = (m^*, g^*, \varepsilon^*) \), the following is acquired

\[
\begin{bmatrix}
\dot{m} \\
\dot{g} \\
\dot{\varepsilon}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial m}{\partial m} |_{s^*} & \frac{\partial m}{\partial g} |_{s^*} & \frac{\partial m}{\partial \varepsilon} |_{s^*} \\
\frac{\partial g}{\partial m} |_{s^*} & \frac{\partial g}{\partial g} |_{s^*} & \frac{\partial g}{\partial \varepsilon} |_{s^*} \\
\frac{\partial \varepsilon}{\partial m} |_{s^*} & \frac{\partial \varepsilon}{\partial g} |_{s^*} & \frac{\partial \varepsilon}{\partial \varepsilon} |_{s^*}
\end{bmatrix}
\begin{bmatrix}
m_t - m^* \\
g_t - g^* \\
\varepsilon_t - \varepsilon^*
\end{bmatrix},
\]

where

\[
J_m =
\begin{bmatrix}
-(1 - \alpha)m^* & 0 & -\frac{\zeta}{\alpha}m^* \\
\left(\frac{\alpha}{\sigma} - 1\right)g^* & g^* & 0 \\
0 & -\varepsilon^* & \frac{(\alpha - \zeta)}{\alpha}\varepsilon^*
\end{bmatrix}.
\]

The characteristic equation associated with \( J_m \) is \( q^3 + c_2q^2 + c_1q + c_0 = 0 \), where

\[ c_0 = \eta \frac{[\sigma(1 - \alpha + \zeta) - \zeta]}{\sigma} m^* g^* \varepsilon^*, \]

\[ c_1 = \eta^2 \frac{\alpha - \zeta}{\alpha} \varepsilon^{*2} - (1 - \alpha)m^* g^*, \]

\[ c_2 = \frac{\alpha - \zeta}{\alpha} \varepsilon^* - \frac{\alpha - \zeta}{\alpha} \varepsilon^* - \frac{\alpha - \zeta}{\alpha} \varepsilon^*.
\]
\[ c_2 = -\eta \frac{(2\alpha - \zeta)}{\alpha} \varepsilon^*. \]

i. Bifurcation Analysis of Uzawa-Lucas Model

This section surveys Barnett and Ghosh’s (2013b) results on the existence of codimension 1 and 2, transcritical, and Hopf bifurcation in the system \((7.6), (7.7),(7.8)\). The codimension, as defined by Kuznetsov (1998), is the number of independent conditions determining the bifurcation boundary. Varying a single parameter helps to identify codimension-1 bifurcation, and varying 2 parameters helps to identify codimension-2 bifurcation. The proof of the following theorem can be found in Barnett and Ghosh (2013b).

**Theorem 7.1** \(\mathbf{J_m}\) has zero eigenvalues, if

\[
\eta \left[ \frac{\sigma(1-\alpha+\zeta)-\zeta}{\sigma} \right] m^* g^* \varepsilon^* = 0. \tag{7.9}
\]

Hopf bifurcations occur at points at which the system has a nonhyperbolic equilibrium with a pair of purely imaginary eigenvalues, but without zero eigenvalues. Also additional transversality conditions must be satisfied. Barnett and Ghosh (2013b) use the following theorem, based upon the version of the Hopf Bifurcation Thereom in Guckenheimer and Holmes (1983):

\(\mathbf{J_m}\) has precisely one pair of pure imaginary eigenvalues, if \(c_0 - c_1 c_2 = 0\) and \(c_1 > 0\). If \(c_0 - c_1 c_2 \neq 0\) and \(c_1 > 0\), then \(\mathbf{J_m}\) has no pure imaginary eigenvalues. The result is:

**Theorem 7.2** The matrix \(\mathbf{J_m}\) has precisely one pair of pure imaginary eigenvalues, if

\[
\begin{align*}
\alpha m^* g^* ((\alpha - 1)\alpha \sigma + \zeta (\sigma - \alpha)) + \eta^2 \sigma \varepsilon^* (2\alpha - \zeta)(\alpha - \zeta) &= 0, \\
\frac{\eta^2}{\alpha} \varepsilon^* (\alpha - \zeta) - (1/\alpha) m^* g^* &> 0.
\end{align*}
\tag{7.10}
\]

ii. Case Studies

Let \(\mathbf{\Phi}^* = \{\eta, \zeta, \alpha, \rho, \sigma, n, \delta\} = (0.05, 0.1, 0.65, 0.0505, 0.15, 0, 0)\) and \(\mathbf{\Phi}^* = \{\eta, \zeta, \alpha, \rho, \sigma, n, \delta\} = (0.05, 0.1, 0.75, 0.0505, 0.15, 0, 0)\). Barnett and Ghosh (2013b) studied the following cases, assuming that free parameters vary at fixed \(\mathbf{\Phi}^*\) (values based on Benhabib and Perli (1994)), while the rest of the parameters are set at \(\mathbf{\Phi}^*\). The following cases give the range of parameter values satisfying conditions (7.9) and (7.10):

Case I: Free parameters, \(\alpha, \eta\).

Case II: Free parameters, \(\zeta, \alpha\).
**Case III**: Free parameters, $\sigma, \alpha$.

**Case IV**: Free parameters, $\zeta, \rho$.

Barnett and Ghosh (2013b) add another parameter as a free parameter and continue with the analysis. The following cases give the range of parameter values satisfying conditions (7.9) or (7.10), while the rest of the parameters are assumed to be at $\omega^*$. Barnett and Ghosh (2013b) studied the following cases:

**Case V**: Free parameters, $\alpha, \zeta, \rho$.

**Case VI**: Free parameters, $\eta, \zeta, \sigma$.

**Case VII**: Free parameters, $\alpha, \eta, \rho$.

**Case VIII**: Free parameters, $\alpha, \sigma, \rho$.

**Case IX**: Free parameters, $\alpha, \eta, \sigma$.

The following is an approach to exploring cyclical behavior in the model. Suppose the externality parameter $\zeta$ increases. This causes the savings rate to increase. This is because when consumers are willing to cut current consumption in exchange for higher future consumptions, in other words, when intertemporal elasticity of substitution for consumption is high ($\sigma$ is low), people start saving more. Hence there is a movement of labor from output production to human capital production. Human capital begins increasing, which implies faster accumulation of physical capital, if sufficient externality to human capital in production of physical capital is present. If people care about the future more (subjective discount rate $\rho$ is lower), consumption starts rising gradually with faster capital accumulation, leading to greater consumption-goods production in the future, which eventually leads to a decline in savings rate. Two opposing effects come into play when savings rate is different from the equilibrium rate, causing a cyclical convergence to equilibrium. Hence, interaction between different parameters can cause cyclical convergence to equilibrium or may cause instability. And for some parameter values we may have convergence to cycles.

Using the numerical continuation package Matcont, Barnett and Ghosh (2013b) further investigate the stability properties of cycles generated by different combinations of parameters. While some of the limit cycles generated by Andronov-Hopf bifurcation are stable (supercritical bifurcation), there could be some unstable limit cycles (subcritical bifurcation) created as well. When Hopf bifurcations are generated, Table 7.1 reports the values of the share of capital ($\alpha$), externality in production of human capital ($\zeta$), and the inverse of intertemporal elasticity of substitution in consumption ($\sigma$). A positive value of the first Lyapunov coefficient indicates creation of subcritical Hopf bifurcation. Thus for each of the cases reported in Table 7.1, an
unstable limit cycle (periodic orbit) bifurcates from the equilibrium. All of these cases also produce branch points (pitchfork/transcritical bifurcations).

Continuation of limit cycles from the Hopf point, when $\alpha$ is the free parameter, gives rise to limit point (Fold/Saddle Node) bifurcation of cycles. From the family of limit cycles bifurcating from the Hopf point, limit point cycle (LPC) is a fold bifurcation, where two limit cycles with different periods are present near the LPC point at $\alpha = 0.738$.

Continuing computation further from a Hopf point gives rise to a series of period doubling (flip) bifurcations. Period doubling bifurcation is defined as a situation in which a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one. The first period doubling bifurcation, at $\alpha = 0.7132369$, has positive normal form coefficients, indicating existence of unstable double-period cycles. The rest of the period doubling bifurcations have negative normal-form coefficients, giving rise to stable double-period cycles.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Equilibrium Bifurcation</th>
<th>Bifurcation of Limit Cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Hopf (H)</td>
<td>Limit Point Cycle (LPC)</td>
</tr>
<tr>
<td></td>
<td>First Lyapunov coefficient = 0.00242, $\alpha=0.738207$</td>
<td>period= 231.206, $\alpha=0.7382042$, Normal form coefficient=0.007</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Period Doubling (PD)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>period= 584.064, $\alpha=0.7132369$, Normal form coefficient=0.910</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Period Doubling (PD)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>period= 664.005, $\alpha=0.7132002$, Normal form coefficient=-0.576</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Period Doubling (PD)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>period= 693.988, $\alpha=0.7131958$, Normal form coefficient=-0.469</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Period Doubling (PD)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>period= 713.978, $\alpha=0.7131940$, Normal form coefficient=-0.368</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Period Doubling (PD)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>period= 725.667, $\alpha=0.7131932$, Normal form coefficient=-0.314</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Period Doubling (PD)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>period= 784.104, $\alpha=0.7131912$, Normal form coefficient=-0.281</td>
</tr>
</tbody>
</table>
The period of the cycle rapidly increases for very small perturbation in the parameter $\alpha$. The limit cycle approaches a global homoclinic orbit. A homoclinic orbit is a dynamical system trajectory, which joins a saddle equilibrium point to itself. In other words, a homoclinic orbit lies in the intersection of equilibrium’s stable manifold and unstable manifold. There exists the possibility of reaching chaotic dynamics in the decentralized version of Uzawa-Lucas model through a series of period doubling bifurcation.

For the cases in which $\zeta$ and $\sigma$ are free parameters, Barnett and Ghosh (2013b) carry out the continuation of limit cycle from the first Hopf point. Both cases give rise to limit point cycles with a nonzero normal form coefficient, indicating the limit cycle manifold has a fold at the LPC point. Similar results are found, but not reported, if the continuation of limit cycles is carried out from the second Hopf point for each of these cases.

### 7.4. Jones Semi-Endogenous Growth Model

First we introduce the model used by Barnett and Ghosh (2013a). The model is based on a variant of Jones (2002) semi-endogenous growth model. The labor endowment equation is given by

$$L_{A_t} + L_{Y_t} = L_t = \varepsilon_t N_t,$$  \hspace{1cm} (7.11)

where at time $t$, $L_t$ is employment, $L_{Y_t}$ is the labor employed in producing output, $L_{A_t}$ is the total number of researchers, and $N_t$ is the total population having rate of growth $n > 0$. Each
A person is endowed with one unit of time and divides the time among producing goods, producing ideas and producing human capital, while $\varepsilon_t$ and $1 - \varepsilon_t$ represent the amount of time the person spends producing output and accumulating human capital, respectively. Physical capital is accumulated by foregoing consumption.

Then

$$\dot{K} = s_k Y_t - dK_t, \quad K_0 > 0,$$

where $s_k$ is the fraction of output invested, $d$ is the exogenous, constant rate of depreciation, $Y_t$ is the aggregate production of homogenous final goods, and $K_t$ is capital stock. Hence, this equation follows:

$$\dot{K} = Y_t - C_t - dK_t.$$  \hspace{1cm} (7.13)

Output is produced using the total quantity of human capital, $H_t$, and a set of intermediaries. Then

$$H_t = h_t L_t,$$  \hspace{1cm} (7.14)

where $h_t$ is human capital per person and $L_t$ is labor employed in producing output. An individual’s human capital, $h_t$, is produced by foregoing time in the labor force. Then

$$\dot{h}_t = \eta h_t^\beta (1 - \varepsilon_t)^\beta - \theta g_A h_t, \quad 0 < \beta_1, \beta_2, \varepsilon_t < 1, \eta > 0, 1 + \theta > 0,$$  \hspace{1cm} (7.15)

where $\eta$ is productivity of human capital in the production of new human capital, $\theta$, reflects the effect of technological progress on human capital investment, and

$$g_A = \frac{\dot{A}}{A}$$

is the growth rate of technology. Equation (7.15) builds on the human capital accumulation equation from the Uzawa-Lucas model (Uzawa(1965) and Lucas(1988)).

The production function is given by

$$Y_t = H_t (1 - \alpha) \int_0^A x(i) \alpha_i, \quad (7.16)$$

where $x(i)$ is the input of intermediate $i$, and $A$ is the number of available intermediates, $\alpha \in (0,1)$, and $\frac{1}{1-\alpha}$ is the elasticity of substitution for any pair of intermediates. Research and development (R&D) enable firms to produce new intermediates. The R&D technology is

$$\dot{A} = \gamma H_t^\lambda A_t^{1-\phi}, \quad \phi > 0, 0 < \lambda \leq 1.$$  \hspace{1cm} (7.17)
According to this equation, new ideas produced at any point in time depend on the effective research effort \((H_A)\) and existing stock of ideas \((A)\), while \(\phi\) represents the externalities associated with R&D. Then

\[ H_A = h_t L_A. \tag{7.18} \]

The representative final output firm rents capital goods, \(x(i)\), from monopolist \(i\) at price \(p(i)\) and pays \(w\) as the rental rate for per unit of human capital. For each durable, the firm chooses a profit-maximizing quantity \(x(i)\) and \(H_y\) to

\[
\max_{x, H_y} \int_0^\infty \left[ H^\alpha_y - p(i) x(i) d_i - w H_y. \right.
\]

Solving the maximization problem gives

\[ p(i) = \alpha H^\alpha_y - x(i) \tag{7.19} \]

\[ w = (1 - \alpha) \frac{y}{H_y}. \tag{7.20} \]

Each intermediate good \(x(i)\) is produced by a monopolist, who owns an infinitely-lived patent on a technology determining how to transform costless a unit of raw material \((K)\) into intermediate good. The production function is \(x = K\). The producer of each specialized durable takes \(p(i)\) as given, from equation (7.19), in choosing the profit maximizing output, \(x\), in accordance with profit level

\[ \pi = \max_x p(x) x - r x, \]

where \(r\) is the rental price of raw capital. Solving the monopoly profit maximization problem gives

\[ p(i) = \bar{p} = \frac{r}{\alpha}. \tag{7.21} \]

The flow of monopoly profit is

\[ \pi(i) = \bar{\pi} = \bar{p} \bar{x} - r \bar{x} = (1 - \alpha) \bar{p} \bar{x}. \tag{7.22} \]

The decision to produce new specialized input depends on a comparison of the discounted stream of net revenue and the cost of the initial investment in a design. Because the market for design is competitive, the price for designs, \(P_A\), will be bid up until equal to the present value of the net revenue that a monopoly can extract.

Hence,
\[
\int_t^\infty e^{-\int_\tau^\infty r(s) \, ds} \pi(\tau) \, d\tau = P_A(t), \quad (7.23)
\]

where \( r \) is the interest rate. Assuming free entry into the R&D sector, the zero profit condition is

\[
wH_A = P_A \gamma H_A^\lambda A^{1-\phi}. \quad (7.24)
\]

If \( v(t) \) denotes the value of the innovation, then

\[
v(t) = \int_t^\infty e^{-\int_\tau^\infty r(s) \, ds} \pi(\tau) \, d\tau. \quad (7.25)
\]

Therefore, equation (14) can equivalently be written as,

\[
wH_A = vY H_A^\lambda A^{1-\phi}. \quad (7.26)
\]

Also because of symmetry with respect to different intermediates, \( K = Ax \). The production function then is

\[
Y = (AH_Y)^{1-\alpha} (K)^\alpha. \quad (7.27)
\]

Hence, from equation (7.20) and (7.27),

\[
w = (1-\alpha)A \left( \frac{K}{AH_Y} \right)^\alpha. \quad (7.28)
\]

From zero profits in the final goods sector, \( \pi = H_Y^{1-\alpha} Ax^\alpha - pAx - wH_Y = 0 \); and from equation (7.20), the following equation results.

\[
Y - wH_Y = pAx = aY. \quad (7.29)
\]

Notice that wages equalize across sectors, as a result of free entry and exit. Individuals maximize intertemporal utility to choose consumption and the fraction of time devoted to human capital production (or the fraction of time devoted to market work). Hence, the agent’s problem is

\[
\max_{c_t, \varepsilon_t} \int_t^\infty e^{-(\rho-n)t} \left[ c(\tau)^{1-\sigma} - 1 \right] 1 - \sigma \, dt
\]

subject to

\[
\dot{K} = r_t[K_t + v_tA_t] + w_t H_t - c_t N_t - v_t A_t - v_t A_t,
\]

\[
\dot{h}_t = \eta h_t^{\beta_1} (1 - \varepsilon_t)^{\beta_2} - \theta g_A h_t, \text{ and } \varepsilon_t \in [0,1].
\]
where $\rho$, with $\rho > n > 0$, is the subjective discount rate, and $\sigma \geq 0$ is the inverse of the intertemporal elasticity of substitution in consumption.

Barnett and Ghosh (2013a) derived the following equations to represent the dynamic equations for the model:

$$\frac{\dot{g}}{g} = \left(\frac{a^2}{\sigma} - 1\right) m - \frac{\rho}{\sigma} + n + g + d,$$

(7.30)

$$\frac{\dot{m}}{m} = \frac{1-\alpha}{\alpha} \left[ -\alpha^2 m + \alpha v + \phi(u - v) \right],$$

(7.31)

$$\frac{\dot{v}}{v} = (1 - \alpha) m + v - g + \left\{ \frac{(1-\alpha)\phi}{\alpha} - 1 \right\} (u - v) - d,$$

(7.32)

$$\frac{\dot{z}}{z} = \frac{1}{f(\beta_2 - 1)} \left[ -z - \theta g_A(\beta_1 - 2) + \alpha v - \beta_2 \frac{zv}{u} - (1 - \phi)(u - v) - n \right] - (1 - \beta_1) (z - \theta g_A),$$

(7.33)

$$\frac{\dot{f}}{f} = \frac{1+f}{f(\beta_2 - 1)} \left[ -z - \theta g_A(\beta_1 - 2) + \alpha v - \beta_2 \frac{zv}{u} - (1 - \phi)(u - v) - n \right],$$

(7.34)

$$\frac{\dot{u}}{u} = z - \theta g_A + n - \phi (u - v) + \frac{1}{f(\beta_2 - 1)} \left[ -z - \theta g_A(\beta_1 - 2) + \alpha v - \beta_2 \frac{zv}{u} - (1 - \phi)(u - v) - n \right].$$

(7.35)

The steady state, $s^* = (g^*, m^*, v^*, z^*, f^*, u^*)$, is such that $\dot{g} = \dot{m} = \dot{v} = \dot{z} = \dot{f} = \dot{u} = 0$, with

$$z^* = \frac{n\theta}{\phi},$$

$$v^* = \frac{\rho - n}{\alpha} + \frac{n\sigma}{\phi \alpha},$$

$$u^* = v^* + \frac{n}{\phi},$$

$$m^* = \frac{v^*}{\alpha} + \frac{n}{\alpha^2},$$

$$g^* = (1 - \frac{a^2}{\sigma}) m^* + \frac{\rho}{\sigma} - n - d,$$

$$f^* = \frac{u^*}{v^* \beta_2} \left( \frac{\phi \rho}{\theta n} - \frac{(\phi + 1 - \sigma)}{\theta} - (\beta_1 - 1) \right).$$

7.4.1. Bifurcation Analysis
Next, the goal is to examine the existence of codimension 1 and codimension 2 bifurcations in the dynamical system defined by (7.30)-(7.35). Varying a single parameter permits us to identify codimension-1 bifurcation and varying 2 parameters permit us to identify codimension-2 bifurcation.

Andronov-Hopf bifurcation is the birth of a limit cycle from an equilibrium in the dynamical system. The equilibrium changes stability through a pair of purely imaginary eigenvalues. Barnett and Ghosh (2013a) used the numerical continuation package Matcont to detect such bifurcations. While some of the limit cycles generated by Hopf bifurcation are stable (supercritical bifurcation), there could be some unstable limit cycles (subcritical bifurcation) created. Table 7.2 reports the values of the subjective discount rate \( \rho \), the share of human capital and the share of time devoted for the human capital production \( \beta_1 \) and \( \beta_2 \) respectively, the effect of technological progress on human capital accumulation \( \theta \), and the depreciation rate of capital \( d \), at which Hopf bifurcation occurs, when those parameters are treated as free parameters.

A positive value of the first Lyapunov coefficient indicates creation of subcritical Hopf bifurcation. Thus for each of the cases reported in Table 7.2, an unstable limit cycle with periodic orbit bifurcates from the equilibrium. When \( \rho, \beta_1, \theta \) and \( d \) are treated as free parameters, a slight perturbation of them gives rise to branch points (pitchfork/transcritical bifurcations).

Barnett and Ghosh (2013a) further investigate the stability properties of cycles generated by different combination of such parameters. Continuation of limit cycles from the Hopf point for the case when \( \rho \) is the free parameter gives rise to two period doubling (flip) bifurcations. Period doubling bifurcation occurs, when a new limit cycle emerges from an existing limit cycle, and the period of the new limit cycle is twice that of the old one. The initial period doubling bifurcations occur at \( \rho = 0.0257 \) and \( \rho = 0.0258 \) with a negative normal form coefficient indicating stable double-period cycles.

Continuing computation further from the Hopf point gives rise to limit point (fold/saddle node) bifurcation of cycles. From the family of limit cycles bifurcating from the Hopf point, limit point cycle (LPC) is a fold bifurcation, where two limit cycles with different periods are present near LPC point at \( \rho = 0.0258 \). Barnett and Ghosh (2013a) find another period doubling (flip) bifurcation upon further computation.

Barnett and Ghosh (2013a) investigate the existence of codimension-2 bifurcations by allowing two free parameters \( \theta \) and \( \rho \) for the first case and \( \theta \) and \( \beta_1 \) for the second. Two points were detected corresponding to codimension 2 bifurcations: Bogdanov-Takens and Generalized Hopf (Bautin) for each of the cases. At each Bogdanov-Takens point the system has an
equilibrium with a double zero eigenvalue. The normal form coefficients \((a, b)\) are reported in Table 7.2 and are all nonzero.

The Generalized Hopf (Bautin) bifurcation is a bifurcation of an equilibrium, at which the critical equilibrium has a pair of purely imaginary eigenvalues, and the first Lyapunov coefficient for the Andronov-Hopf bifurcation vanishes. The bifurcation point separates branches of subcritical and supercritical Andronov-Hopf bifurcations in the parameter plane. The Generalized Hopf points are nondegenerate, since the second Lyapunov coefficient \(L_2\) is nonzero. For nearby parameter values, the system has two limit cycles, which collide and disappear through a saddle–node bifurcation.

**Table 7.2**

<table>
<thead>
<tr>
<th>Parameters varied</th>
<th>Equilibrium bifurcation</th>
<th>Continuation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_1: {a = 0.4, \rho = 0.055, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 8})</td>
<td>Branch Point (BP) (\beta_1 = 1)</td>
<td></td>
</tr>
<tr>
<td>(\beta_2: {a = 0.4, \rho = 0.025772, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08})</td>
<td>Hopf (H) (\beta_1 = 0.19)</td>
<td></td>
</tr>
<tr>
<td>(d: {a = 0.4, \beta_1 = 0.19, \rho = 0.055, \beta_2 = 0.04, n = 0.01, \theta = 0.4, \phi = 1, \sigma = 8})</td>
<td>Hopf (H) (\beta_2 = 0.040000)</td>
<td></td>
</tr>
<tr>
<td>(\rho: {a = 0.4, \beta_1 = 0.19, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 0.08})</td>
<td>Hopf (H) (d = 0.826546)</td>
<td></td>
</tr>
<tr>
<td>(\theta: {a = 0.4, \beta_1 = 0.19, \beta_2 = 0.04, n = 0.01, d = 0, \rho = })</td>
<td>Hopf (H) Neutral saddle (\rho = 0.026698)</td>
<td></td>
</tr>
<tr>
<td>(\beta_1: {a = 0.4, \rho = 0.025772, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08})</td>
<td>Branch Point (BP) (\beta_1 = 1)</td>
<td></td>
</tr>
<tr>
<td>(\beta_2: {a = 0.4, \rho = 0.025772, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 0.8, \sigma = 0.08})</td>
<td>Hopf (H) (\beta_2 = 0.040000)</td>
<td></td>
</tr>
<tr>
<td>(d: {a = 0.4, \beta_1 = 0.19, \rho = 0.055, \beta_2 = 0.04, n = 0.01, \theta = 0.4, \phi = 1, \sigma = 8})</td>
<td>Hopf (H) (d = 0.826546)</td>
<td></td>
</tr>
<tr>
<td>(\rho: {a = 0.4, \beta_1 = 0.19, \beta_2 = 0.04, n = 0.01, d = 0, \theta = 0.4, \phi = 1, \sigma = 0.08})</td>
<td>Hopf (H) (\beta_2 = 0.040000)</td>
<td></td>
</tr>
<tr>
<td>(\theta: {a = 0.4, \beta_1 = 0.19, \beta_2 = 0.04, n = 0.01, d = 0, \rho = })</td>
<td>Hopf (H) Neutral saddle (\rho = 0.026698)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Bifurcation of limit cycle</th>
<th>Period doubling (period=1569.64; (\rho = 0.0257))</th>
<th>Normal form coefficient=-4.056657e-013</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period doubling (period=1741.46; (\rho = 0.0258))</td>
<td>Normal form coefficient=-7.235942e-015</td>
<td></td>
</tr>
<tr>
<td>Limit point cycle (period=2119.53; (\rho = 0.0258))</td>
<td>Normal form coefficient=7.894415e-004</td>
<td></td>
</tr>
<tr>
<td>Period doubling (period=2132.13; (\rho = 0.0258))</td>
<td>Normal form coefficient=-1.763883e-013</td>
<td></td>
</tr>
</tbody>
</table>
0.029710729, φ = 0.69716983, σ = 0.08)

First Lyapunov coefficient = 0.00001973, θ = 0.355216

<table>
<thead>
<tr>
<th>Hopf (H)</th>
<th>Neutral saddle θ = 0.612624</th>
</tr>
</thead>
<tbody>
<tr>
<td>Branch Point (BP)</td>
<td>θ = 0.613596</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Codimension-2 bifurcation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized Hopf (GH)</td>
</tr>
<tr>
<td>θ = 0.000064, ρ = 0.580853, L2 = 0.000001254</td>
</tr>
<tr>
<td>Bogdanov-Takens (BT)</td>
</tr>
<tr>
<td>θ = 0, ρ = 0.644247 (a, b) = (0.000001642, -0.003441)</td>
</tr>
</tbody>
</table>

8. Zellner’s Marshallian Macroeconomic Model

8.1. Introduction

The Marshallian Macroeconomic Model (MMM) in Zellner and Israilevich (2005) provides a novel way to examine sectoral dynamics through the introduction of a dynamic entry/exit equation in addition to the usual demand and supply function found in models of this class. The entry/exit behavior model in the MMM can be described by the equation \( \frac{N}{N} = \gamma'(Π - F^e) \); i.e. the growth rate of firms in the industry is proportional to the difference in current industry profitability, \( Π \), and the long-run future profitability in the industry, \( F^e \). The speed of adjustment is determined by the parameter, \( γ' \). Zellner and Israilevich (2005) describe the emergence of rich dynamics in key variables, such as price and output at the sectoral, as well as at the aggregate, level once an entry/exit equation for each industry is introduced into the model. In the simulation exercises conducted by Zellner and Israilevich (2005), \( γ' \) and \( F^e \) were fixed parameters. Varying these parameters would change the equilibria and could possibly cause changes in the nature of the equilibria, such as the number of solutions and the stability properties of the equilibria. Banerjee, Barnett, Duzhak, and Gopalan (2011) undertake the task of examining the model’s characteristics, as well as the possibility of cyclical behavior with respect to the entry/exit parameter \( F^e \) by searching for a bifurcation within the theoretically feasible parameter space in the Marshallian Macroeconomic Model.

Banerjee, Barnett, Duzhak, and Gopalan (2011) showed that a Hopf bifurcation exists within the theoretically feasible parameter space, giving rise to stable cycles. The choice of \( F_1 \) as the candidate for bifurcation parameter re-emphasized the importance of a dynamic entry/exit equation in models of this class. There are several avenues that need to be explored in future work. One such possibility would be the introduction of expectations in firms’ future
profitability. Another one is to introduce the money market and examine the possibility of other kinds of bifurcations with respect to government and monetary policy parameters.

### 8.2. The model

Banerjee, Barnett, Duzhak, and Gopalan (2011) consider a two sector, continuous time version of the Marshallian Macroeconomic Model (MMM), as outlined in Zellner and Israilevich (2005). Each sector is characterized by an aggregate demand function for its output, an aggregate supply function, and aggregate input demand functions for labor and capital. Banerjee, Barnett, Duzhak, and Gopalan (2011) also include the government that collects taxes on output, purchases output from the two sectors, and inputs from the factor markets. They exclude the presence of money markets from the model at this stage to make our analysis simpler.

As in Zellner and Israilevich (2005), these demands are given exogenously, but some of the other factors determining household demand are omitted for simplicity. Aggregate demand is thus given by

\[ S_i = G_i + P_i \frac{1}{1-\eta_{ii}} P_j^{\eta_{ij}} (S(1-T^s))^{\eta_{is}}, \]  

(8.1)

where \( G_i \) is the nominal government expenditure in sector \( i \), \( S = S_1 + S_2 \) is the total income (nominal output), \( T^s \) is the tax rate, \( \eta_{ii} \) is the own price elasticity, \( \eta_{ij} \) is the cross price elasticity, and \( \eta_{is} \) is the income elasticity. Expressed in terms of growth rates, the aggregate demand for goods in each sector is the weighted sum of growth rates of demand from the government and households

\[ S^\prime_i = g_i \hat{G}_i + (1-g_i) \left[ (1-\eta_{ii}) \hat{P}_i + \eta_{ij} \hat{P}_j + \eta_{is} (\hat{S} + \hat{T}^s) \right], \]

(8.2)

where \( g_i \) is the ratio of government spending in sector \( i \) to total sales in sector \( i \) and \( T^{is} = 1 - T^s \).

There are \( N_i \) identical firms in the \( i \)th sector, each using a Cobb-Douglas type production function, \( q_i = A_i L_i^\alpha K_i^\beta \), with \( 0 < \alpha_i, \beta_i < 1 \) and \( 0 < \theta_i = 1 - \alpha_i - \beta_i < 1 \), where \( q_i \) is the product of a neutral technological change, labor and capital augmentation factors. The aggregate nominal profit-maximizing output supply of each sector, \( i \), is the number of firms in the sector, \( N_i \), times the nominal profit-maximizing supply of each firm within that sector. This is given by

\[ S_i = N_i P_i \omega^{-\frac{\alpha_i}{\alpha_i r}} - \frac{P_i}{\theta_i}. \]

Expressing this in growth rates, we have

\[ S^\prime_i = N_i + \frac{1}{\theta_i} \hat{P}_i - \frac{\alpha_i}{\theta_i} \hat{\omega} - \frac{\beta_i}{\theta_i} \hat{P}_i. \]

(8.3)
Banerjee, Barnett, Duzhak, and Gopalan (2011) consider the simplest form of the entry/exit equation proposed by Zellner and Israilevich (2005),

\[ \hat{N}_i = \gamma_i [\Pi_i - F_i], \]  

(8.4)

where \( \Pi_i = \theta_i S_i \) is the current nominal aggregate industry profit for sector \( i \), while \( F_i \) represents the aggregate long-run equilibrium profits in sector \( i \), taking account of discounted entry costs. The parameters \( F_i \) and \( \gamma_i \) are both positive. In the work of Zellner and Israilevich (2005), Banerjee, Barnett, Duzhak, and Gopalan (2011) consider those parameters to be time invariant. The coefficient, \( \gamma_i \), is the speed of adjustment for sector \( i \). Given that \( \gamma_i \) is assumed to be positive, Banerjee, Barnett, Duzhak, and Gopalan (2011) can interpret the entry/exit equations as follows. A positive departure from equilibrium profits, \( F_i^e \), will attract new firms into the industry, while a negative departure will induce firms to leave the industry. The larger the value of \( \gamma_i \), the faster will be this adjustment.

Total nominal, government expenditure, \( G \), is the sum of expenditures in each of the two sectors, \( G_i \), and its expenditure on labor, \( G_L \), and capital, \( G_K \). Zellner and Israilevich (2005) assume that each component of government expenditure, \( G_i \), for all \( i = 1, 2, L, K \), grows at the same rate as the total government expenditure. Banerjee, Barnett, Duzhak, and Gopalan (2011) accommodate this assumption by proposing that \( G_i = \zeta_i G \), where \( \zeta_i \) is the fraction of total government expenditure in the \( i \)th market. Thus in terms of growth rates, we have \( \hat{G}_i = \hat{G} \).

The government collects a single uniform tax at the rate \( T^s \) on output. The tax revenue, \( R \), is thus given by \( R = T^s \cdot S \), which expressed in terms of growth rates yields \( \hat{R} = \hat{T}^s + \hat{S} \). Zellner and Israilevich (2005) further assume that there is an exogenously determined deficit/surplus, \( D \), defined as the government expenditures as a percentage of revenues, so that \( D = \frac{G}{R} \). Thus the flow budget identity of the government in terms of growth rates is

\[ \hat{G} = \hat{D} + \hat{R} = \hat{D} + \hat{T}^s + \hat{S}. \]  

(8.5)

Given the Cobb-Douglas technologies, the aggregate, profit-maximizing factor demands from sector \( i \) are \( L_i = \frac{a_i S_i}{\omega} \) and \( K_i = \frac{b_i S_i}{r} \). The government demand for labor and capital are \( L_g = \frac{G_L}{\omega} \) and \( K_g = \frac{G_K}{r} \), respectively. The total demand for each factor in terms of growth rates is then the weighted sum of growth rates of sectoral demands and the government demand for that factor, as stated below in (8.6) and (8.7).

\[ \frac{L_1}{L} \hat{L}_1 + \frac{L_2}{L} \hat{L}_2 + \frac{L_g}{L} \hat{L}_g = l_1 \hat{L}_1 + l_2 \hat{L}_2 + l_g \hat{L}_g, \]  

(8.6)

\[ \frac{K_1}{K} \hat{K}_1 + \frac{K_2}{K} \hat{K}_2 + \frac{K_g}{K} \hat{K}_g = k_1 \hat{K}_1 + k_2 \hat{K}_2 + k_g \hat{K}_g, \]  

(8.7)
The explicit dependence of the weights, \( l_i \) and \( k_i \), on \( S_1 \) and \( S_2 \) is given in Appendix A in Banerjee, Barnett, Duzhak and Gopalan (2011).

The supply of factors of production is again assumed to be exogenous, as in Zellner and Israilevich (2005) with \( L = \left( \frac{\omega}{p} \right)^{\delta} \left( \frac{\hat{S}}{p} \right)^{\delta_s} \) and \( K = \left( \frac{\gamma}{p} \right)^{\phi} \left( \frac{\hat{S}}{p} \right)^{\phi_s} \), where \( \delta \) (resp. \( \phi \)) and \( \delta_s \) (resp. \( \phi_s \)) are price and income elasticities of labor (resp. capital). In terms of growth rates, the labor and capital supplies equal

\[
\hat{L} = \delta(\hat{\omega} - \hat{P}) + \delta_s(\hat{S} - \hat{P}), \quad (8.8)
\]

\[
\hat{K} = \phi(\hat{\gamma} - \hat{P}) + \phi_s(\hat{S} - \hat{P}). \quad (8.9)
\]

The growth rate of aggregate nominal sales and the price aggregate are given by

\[
\hat{S} = s_1\hat{S}_1 + s_2\hat{S}_2, \quad (8.10)
\]

\[
\hat{P} = s_1\hat{P}_1 + s_2\hat{P}_2, \quad (8.11)
\]

where \( s_i = \frac{S_i}{S} \).

The above model is solved using market clearing conditions in all markets, that is output and factor markets, and incorporating the government’s flow budget identity. The complete solution procedure is outlined in Appendix A in Banerjee, Barnett, Duzhak and Gopalan (2011). They are able to reduce all these equations to yield the following two dynamic equations, which govern the behavior of \( S_1 \) and \( S_2 \):

\[
\begin{bmatrix}
\dot{S}_1 \\
\dot{S}_2
\end{bmatrix} = \begin{bmatrix}
F_1(S_1, S_2; \Omega) \\
F_2(S_1, S_2; \Omega)
\end{bmatrix} = \mathbf{F}(S_1, S_2; \Omega). \quad (8.12)
\]

The explicit form of the non-linear functions \( F_1 \) and \( F_2 \) can be found in Appendix A in Banerjee, Barnett, Duzhak and Gopalan (2011). The vector \( \Omega \) consists of all structural parameters. Banerjee, Barnett, Duzhak, and Gopalan (2011) consider \( F_1 \), the entry parameter for sector 1, as their bifurcation parameter, when they look for a co-dimension 1 bifurcation in the following section.

An equilibrium for this model would constitute a value of \((S_1, S_2)\), at which \( \dot{S}_1 = 0 \) and \( \dot{S}_2 = 0 \), i.e. in the system (8.12), \( \mathbf{F}(S_1, S_2; \Omega) = 0 \).

\[
\mathbf{F}(S_1, S_2; \Omega) = (\mathbf{H}(S_1, S_2; \Omega))^{-1}\mathbf{D}(S_1, S_2; \Omega), \quad (8.13)
\]

where \( \mathbf{H} \) is a matrix of dimension \( 2 \times 2 \) and \( \mathbf{D} \) is a vector of dimension \( 2 \times 1 \). The elements of \( \mathbf{H} \) and \( \mathbf{D} \) are an indication of the high degree of nonlinearity involved in determining the
dynamics of the above mentioned equilibrium. As expected, there will be several equilibria, which can arise due to this non-linearity in $\mathcal{F}$.

However, it is easy to see from equation (8.13) that the values of $S_1$ and $S_2$, at which $\mathcal{D} = 0$, will always be an equilibrium. Under the assumption that there is no growth in government deficit ($D$) and taxes ($T^s$), this solution directly corresponds to the solution of the entry/exit equation given in (8.4), so that

$$S_1 = \frac{1}{\theta_1} F_1 \text{ and } S_2 = \frac{1}{\theta_2} F_2$$

(8.14)

These solutions are economically relevant, since they are positive and ensure that there is no further entry/exit in both sectors, implying a long run equilibrium. The next section surveys Banerjee, Barnett, Duzhak, and Gopalan (2011) results on stability and their bifurcation analysis of this equilibrium.

### 8.3.1. Stability and Bifurcation Analysis of Equilibrium

The dynamics in this two sector MMM, in terms of convergence to the equilibrium given (8.14), can be described by generalizing the analysis of the one sector MMM by Veloce and Zellner (1985). Using the entry/exit equation assumed in Zellner and Israilevich (2005) and solving the one sector model as in Veloce and Zellner (1985) would yield the following differential equation, which is equivalent to (8.12) for the one sector model:

$$\dot{S} = a S (S - F),$$

where $a$ depends on the structural parameters. Here the stationary solutions $S = F$ is stable, if and only if $a < 0$, which is true, if and only if demand is inelastic. Suppose $S > F$, so that current profitability is greater than equilibrium profitability. Then firms will enter, causing the market supply to increase, resulting in a lower price. This drop in price will result in a lower aggregate sales, $S$, (because of the inelasticity of demand), decreasing the difference between the current profitability and equilibrium profitability. With the one sector dynamics governed by the logistic function given above, this process will result in a monotonic path to the equilibrium in a continuous time model. On the other hand, if demand is elastic, then the solution is unstable and any deviation from the equilibrium will result in divergence.

However, in a multisector model, Banerjee, Barnett, Duzhak, and Gopalan (2011) need to consider the effects of cross price and income elasticities along with own price elasticity. Two interesting features arise with respect to the equilibrium dynamics in the multisector model. First, unlike the one sector model, even when the two sectors have elastic demand (own price elasticity greater than 1), the solution may be stable. Second, the path to the long run equilibrium may not be monotonic, so it may depict oscillatory behavior.
There is an oscillatory convergence to equilibrium. It is worth mentioning that this delicate mechanism depends crucially on the own price, cross price, and income elasticities, and the magnitude of shifts in demand and supply in each sector. It is definitely possible that these shifts are not sufficient and may result in the solution being unstable.

Banerjee, Barnett, Duzhak, and Gopalan (2011) emphasize that for an oscillatory convergence, the elasticity parameters need to be consistent with values of the other parameters in production, input markets, entry/exit equations, and government policy. If some of these parameters were to change, then it is very likely that the economy may go from cyclical convergence to persistent cycles or even explosive behavior. In order to investigate this possibility in the MMM, we look for a bifurcation within the theoretically feasible parameter space.

Banerjee, Barnett, Duzhak, and Gopalan (2011) examine the existence of a Hopf bifurcation of codimension-1. In the following analysis, Banerjee, Barnett, Duzhak, and Gopalan (2011) vary only parameter $F_1$, while all other parameters will be maintained at their theoretically feasible values given in Appendix B in Banerjee, Barnett, Duzhak, and Gopalan (2011). This procedure of varying a single parameter helps to identify a codimension-1 bifurcation. In particular, Banerjee, Barnett, Duzhak, and Gopalan (2011) investigate the presence of a Hopf bifurcation, which occurs when the Jacobian of $\mathcal{F}$ has a pair of purely imaginary eigenvalues at some critical value of the parameter $F_1$.

In order to analyze a codim-1 Hopf bifurcation for the system (8.12), Banerjee, Barnett, Duzhak, and Gopalan (2011) first look for the value of $(S_1, S_2)$ and the bifurcation parameter ($F_1$) at which the following conditions hold simultaneously:

$$\mathcal{F}_1(S_1, S_2, F_1) = 0,$$
$$\mathcal{F}_2(S_1, S_2, F_1) = 0,$$
$$tr(J_F(S_1, S_2, F_1)) = 0,$$
$$det(J_F(S_1, S_2, F_1)) > 0,$$

where $J_F$ is the Jacobian of $\mathcal{F}$. Equations (8.15) and (8.16) yield the equilibrium for the system of differential equations in (8.12). In particular, Banerjee, Barnett, Duzhak, and Gopalan (2011) pay attention to the solution given in (8.14). Conditions (8.17) and (8.18) are sufficient to ensure that the eigenvalues of $J_F$ are purely imaginary. It is clear that at the computed critical value $F^H = 6.070386762$, conditions (8.17) and (8.18) are satisfied and the slope of the trace is not zero, implying a Hopf bifurcation. Thus as the parameter $F_1$ crosses $F^H$ from the right, the solution given in (8.14) goes from a stable equilibrium to an unstable one. In fact, the system is
locally spiraling inward for $F_1 > F^H$, and for $F_1$ close enough to $F^H$ and $F_1 < F^H$, the system exhibits stable cycles in the phase space.

9. Conclusion

At this stage of this research, we believe that Grandmont’s conclusions appear to hold for all categories of dynamic macroeconomic models, from the oldest to the newest. So far, the findings we have surveyed suggest that Barnett and He’s initial findings with the policy-relevant Bergstrom-Wymer model appear to be generic. We anticipate that further studies with other models will produce the same results, and advances in nonlinear and stochastic bifurcation are likely to find even deeper classes of bifurcation behavior, including perhaps chaos, which is precluded by linearization. This survey is designed to facilitate such future studies.

The practical implications of these findings include the following. (1) Policy simulations with macroeconometric models should be run at various points within the confidence regions about parameter estimates, not just at the point estimates. Robustness of dynamical inferences based on simulations only at parameter point estimates are suspect. (2) Increased emphasis on measurement of variables is warranted, since small changes in variables can alter dynamical inferences by moving bifurcation boundaries and their distances from parameter point estimates. (3) While bifurcation phenomena are well known to growth model theorists, econometricians should take heed of the views of systems theorists, who have found that bifurcation stratification of the parameter space of dynamic systems is normal, and should not be viewed as a source of model failure or defect.
References


