Sunspots and Lotteries in Growth Economies *

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Abstract

We investigate the relation between lotteries and sunspot allocations in a dynamic economy where the utility functions are not concave. In an intertemporal competitive economy, the household consumption set is identified with the set of lotteries, while in the intertemporal sunspot economy it is the set of measurable allocations in the given probability space of sunspots. Sunspot intertemporal equilibria whenever they exist are efficient, independently of the sunspot space specification. If feasibility is, at each point in time, a restriction over the average value of the lotteries, competitive equilibrium prices are linear in basic commodities and intertemporal sunspot and competitive equilibria are equivalent. Two models have this feature: Large economies and economies with semi-linear technologies. We provide examples showing that in general, intertemporal competitive equilibrium prices are non-linear in basic commodities and, hence, intertemporal sunspot equilibria do not exist.

The competitive static equilibrium allocations are stationary, intertemporal equilibrium allocations, but the static sunspot equilibria need not to be stationary, intertemporal sunspot equilibria. We construct examples of non-convex economies with indeterminate and Pareto ranked static sunspot equilibrium allocations associated to distinct specifications of the sunspot probability space. Furthermore, we show that there exist large economies with a countable infinity of Pareto ranked static sunspot equilibrium allocations with aggregate pro capita consumption invariant both across equilibria as well as across realizations of the uncertainty. This implies that these equilibria are equilibria of a pure exchange economy with non-convex preferences.

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1 Introduction

Two views of Sunspot Economies

The sunspot literature, originated by the Cass and Shell, [3], studied convex economic environments where the First Welfare Theorem does not hold. In these economies the introduction of "extrinsic uncertainty" may change the set of equilibrium allocations. Indeed the last assertion became a Folk theorem, [15]. Most notably, there are economies where Pareto efficient *certainty* equilibria (i.e., equilibria where sunspots do not matter) coexist with inefficient sunspot equilibria (e.g., [1]). This first research saw sunspot as a socially undesirable, but possible, and to some extent, pervasive, equilibrium outcome of competitive environments.

In the recent years, research on sunspots has taken quite a different route. The objects of study are economies with non-convexities (see for instance [10]). It was soon clear that non convexities in production or in preferences had very different implications.

When non-convexities are present only in the production sets of the firms, the introduction of sunspots is immaterial. Given the linearity of the profit functions, a sunspot equilibrium is an equilibrium of the "concavified" economy and, thus, sunspots do not matter [2].

The situation is very different when non-convexities are present in the consumers' preferences or consumption set. Most of the literature deals with static economies with non convexities of the consumption set, generated by either indivisibilities, [16], or incentive compatibility constraints, [8]. In these environments randomness in allocations may improve welfare. The latter can be generated in two different ways, either by introducing lotteries or by introducing sunspots. In the sequel, we use the term *competitive equilibrium* to refer to the equilibrium outcome of an environment where individuals face a complete set of markets for lotteries and the term *sunspot equilibrium* when they face, for given specification of the probability space of extrinsic uncertainty, a complete set of (sunspots) contingent markets.

Competitive Equilibria and Sunspot Equilibria

Two major differences between the concept of competitive equilibrium and sunspot equilibria are clear, and both of them are crucial. The first difference concerns the nature of the consumption sets. With the competitive notion, agents may choose any probability distribution over basic consumption bundles, as in [11]. With the sunspot notion, they choose measurable maps with respect to an exogenously given probability space of extrinsic uncertainty $(\Omega, \mathfrak{A}, \hat{\sigma})$.

The second difference concerns the nature of prices in the two concepts. In the lottery economy prices are linear in lotteries and, hence, they can be non-linear in commodities. In the sunspot economy, prices are linear in sunspot contingent allocations and, hence, they must be linear in contingent commodities. Thus, a first natural question is to investigate whether competitive and sunspot equilibria are "equivalent." There are two aspects to this question. The first is at the level of individual behavior and it concerns the different nature of the consumption sets under the two specifications. The second is at an equilibrium level and it concerns the nature of equilibrium prices. Any individual sunspot allocation induces a probability distribution over consumption bundles and, hence, it induces a lottery. However, the viceversa need not be true. A lottery induces a stochastic allocation, but this may be either not measurable with respect to the given probability space of extrinsic uncertainty, $(\Omega, \mathfrak{A})$, or incompatible with the given sunspot probability law, $\hat{\sigma}$. When each lottery induces a (compatible) sunspot allocation, lotteries and sunspot allocations are equivalent. Garret, Kreister, Qui and Shell [5] show that equivalence is achieved by any "continuous randomizing device"[1]. When the probability space of extrinsic uncertainty is not rich enough, sunspot equilibrium allocations, if they exist, may be Pareto suboptimal. Different specifications of $(\Omega, \mathfrak{A}, \hat{\sigma})$ may yield different equilibria with, in principle, Pareto ranked allocations ([16], [4]). This is not a contradiction of the First Fundamental Theorem of Welfare Economics. Pareto optimality is relative to a given consumption set, while in the previous statements we are comparing allocations in different consumption sets. [2]

However, when the extrinsic uncertainty space is $([0,1), B, L)$, under fairly general assumptions, every sunspot equilibrium allocation induces an equilibrium lottery allocation. If equilibrium allocations of the lottery economy are supported by linear prices, the viceversa is also true, ([5] and [6]). This equivalence result is based on the assumed existence of sunspot equilibria and, equivalently, of lottery equilibria supported by linear prices.

Thus we are led to the second aspect of the problem: singling out environments where competitive equilibrium prices are linear. In any economy where feasibility is a restriction on the average values of (joint) lotteries, equilibrium prices may be restricted (without loss of generality) to be linear. One such environment is a large pure exchange economy (with finitely many types). If each household of type i consumes iid lotteries β_i, by the law of large numbers, the aggregate average allocation is, with probability 1, deterministic and equal to the average (appropriately taken over types and lotteries) consumption. Feasibility in a large exchange economy requires that average endowment to be equal to average consumption. Hence, feasibility is just a restriction on the average value of the lotteries. Thus, in large economies lottery and sunspot equilibrium allocations are equivalent, [8] and [5].

The model

We depart from the existing literature in two aspects. First, the lack of convexity is generated by the *preferences* of the households and not by the nature of their

[1] For instance, the probability space $([0,1), B, L)$, with B denoting the Borel subsets of $[0,1)$ and L the Lesbegue measure.

[2] In this literature as well as in this paper, a feasible (sunspot) allocation is efficient if it does not exists a feasible lottery that Pareto dominates it. Furthermore, it is understood that preferences are extended over random allocations via expected utility.

consumption set. In previous work, [12], we have studied the Lancaster model of characteristics, [9]. When the household production function transforming commodities into characteristics is concave, but not linear, the derived utility function over commodities (i.e., the composition of the concave utility function over characteristics with the concave production function) can be any continuous function. Furthermore, a long history of experimental evidence indicates the lack of linearity of the household production function. Thus, the Lancaster model of characteristics provides a sound justification for the study of economies with non-convex preferences.

The second key feature of our model is time. We study a continuous time growth economy. In this economy lotteries appear naturally because they are the only sensible way to define the limit of allocation paths (over time) that are in the limit achieving the supremum in the growth problem. Since the utility function is not concave, an optimal deterministic path may not exist: however, the sequence of paths yielding the supremum value is oscillating among extreme points. The natural limit of these paths is a path taking values in lotteries. Hence, in our model, the household consumption set is identified with the set of lotteries and the feasibility requirement equates the effective investment to the average production of investment goods at each instant of time. The definitions of the consumption set and of the feasibility requirement are, in an obvious sense, the outcome of the optimal use of time as a convexifying device. The same type of mechanism is at work for competitive versions of these growth economies. Households facing a budget constraint reproduce, by varying their consumption *over time,* the distribution of their consumption over goods induced by a lottery. In a way they would, by their intertemporal choices, extend the commodity space. This extension is precisely our "natural" choice of commodity space.

A Growth Economy

The growth economy provides thus a natural environment to study the relation between lotteries and sunspot allocations. We examine this relation by using several specifications of the economy, differentiated by time characteristics and the number of households. The competitive equilibrium describes the growth economy where the household consumption set and the feasibility requirement are as previously described and there exist complete markets for time contingent lotteries. In a sunspot equilibrium, for given (time invariant) sunspot probability space $(\Omega, \mathfrak{A}, \hat{\sigma})$, the consumption set of the individuals is identified, at each instant of time, with the set of $(\Omega, \mathfrak{A})$-measurable contingent commodity allocations. Feasibility, conformably with the notion of competitive equilibrium, requires, at each instant of time, the effective investment to be equal to the average production of investment goods. The average, this time, is taken over the realization of the sunspot allocation according to the sunspot probability measure $\hat{\sigma}$.

In a (finite) pure exchange economy with a representative agent, feasibility requires the sunspot allocations to be equal to the sunspot invariant endowments. Thus, sunspots do not matter. However, in the growth economy, the adopted notion of feasibility allows for non trivial randomness of feasible allocations even in the model with a single representative consumer.

An equilibrium can be either intertemporal or static and the economy can be with a single representative household or with a large number of identical ones. The last distinction is immaterial in convex economies, but has important consequences when preferences display non-convexities.

The intertemporal economy is just the standard competitive version of the growth economy. We compare the intertemporal competitive equilibrium allocations with the intertemporal sunspot equilibrium allocations. If, at each competitive equilibrium, lotteries are priced linearly, a quite strong result holds true: competitive allocations and sunspot allocations coincide. Also, without loss of generality, probability adjusted prices are sunspot invariant.

Most importantly and in the contrast with the literature, independently of the sunspot space specification, there cannot exists inefficient sunspot equilibria. The reason is quite simple. Households can use time. If a sunspot probability space does not provide enough randomness to reproduce the desired lotteries, time supplies the additional missing noise. This in turn implies that it cannot exists a solution to the household programming problem in the sunspot economy or, equivalently, that there cannot be an intertemporal sunspot equilibrium for "inadequate" specifications of $(\Omega, \mathfrak{A}, \hat{\sigma})$. Hence, the key question is to single out environments guaranteeing price linearity of the lotteries in basic commodities. As already argued, the large economy is always one of such environments. The second requires a non-generic restriction on the technology that we call semi-linearity. With a semi-linear technology, for each given vector of capital goods, the isoquant curves of investment and consumption goods are linear (as in the text book growth model). In an economy with semi-linear technologies, a lottery is feasible if its average value is feasible. The latter is the key desideratum to obtain linear supporting prices.

However, with a representative household (or with finitely many), for general specifications of the technology, lottery prices are non-linear and sunspot equilibria do not exists. We provide examples of this fact by using the static version of the economy. The notion of static equilibrium is the restriction of the definition of intertemporal equilibrium to vectors of allocations and prices that are constant over time. Thus, each static competitive (lottery) equilibrium is a stationary, intertemporal competitive equilibrium and viceversa.

At a static equilibrium, economic activity takes one period. Households maximize utility by selecting a budget feasible lottery, in a static competitive equilibrium, or a budget feasible $(\Omega, \mathfrak{A})$ -measurable allocation, in a static sunspot equilibrium. The wealth of the household(s) coincides with the value of the firm net of capital expenses. The firm maximizes profits by selecting output (lotteries or measurable allocations, depending on the economic environment), capital stocks and investment goods. As for the intertemporal economy, the firm production set requires the effective investment goods to be equal to the average value of their production. The link between the intertemporal and the static equilibria is provided by an appropriate definition of the cost capital and by the requirement that at equilibrium investment goods are equal to zero. The allocation and price vector of a static equilibrium solve the Hamiltonian system for a stationary intertemporal allocation of the optimal growth problem. Thus, by the Second Fundamental Theorem of Welfare Economics, the competitive static equilibrium allocations are stationary, intertemporal equilibrium allocations.

Stationary allocations are important, simple to analyze and they are constructed, in standard economies, using our notion of static equilibria. Indeed in standard, convex environments, static and stationary intertemporal equilibria coincide. However, in our environment, sunspot (static) allocations behave differently. There are non-convex economies with indeterminate and Pareto ranked static sunspot equilibrium allocations associated with distinct specifications of probability space $(\Omega, \mathfrak{A}, \sigma)$. By what previously said, inefficient static sunspot allocations can never be stationary, intertemporal sunspot equilibrium allocations. Thus, when preferences are not convex, static sunspot equilibria are an inadequate tool to analyze stationary equilibrium allocations of growth economies. Furthermore, we show that there exists large economies with a countable infinity of Pareto ranked static sunspot equilibrium allocations with aggregate pro capita consumption invariant both across equilibria as well as across realizations of the uncertainty. Given the definition of the static sunspot equilibrium, these equilibria are equilibria of a large pure exchange economy with non-convex preferences. Thus, although of no consequence for dynamic economies, this construction shows how problematic the relation between sunspot and competitive equilibria is for pure exchange economies with non-convexities in preferences.

2 The Representative Household Economy

We model a growth economy with a representative household. The model is standard except for the lack of convexity of the household's preferences. The particular notions of competitive and sunspot equilibria that we use are much related to the solution of an optimal growth problem with, potentially, a non-concave utility function. Hence, in order to clearly explain the various adopted definitions of equilibrium we turn next (and quickly) to the optimal growth problem. A detailed analysis of the growth problem is in Rustichini and Siconolfi ([13]).

2.1 The optimal growth problem

Consider a growth economy with a representative individual. The consumption set in the basic commodity space is Y, a non-empty, compact and convex subset of $\Re^C_+$. The instantaneous utility function, $U : Y \to \Re$, is continuous, but not necessarily concave. A vector of capital goods is $x \in \Re^K_+$. The continuous and jointly concave function $F : \Re^K_+ \times Y \to \Re^K_+$ describes the technology. Let $\hat{y} : Y \to \Re_+$ denote a measurable map, $\hat{y} = (\hat{y}(t))_{t \geq 0}$. For given initial condition $\hat{x}(0) = x_0$, the optimal growth problem is:

$$\sup_{\hat{y}} \int_0^\infty e^{-rt} U(\hat{y}(t)) dt \tag{1}$$

subject to the constraint:

$$\dot{x}(t) = F(\hat{x}(t), \hat{y}(t)),\ \hat{x}(0) = x_0,\ \hat{y}(t) \in Y. \tag{2}$$

If U is not concave, an optimal solution to 1 may not exist.

The reason is simple. If preferences are not convex, a consumption path (that is, the $\hat{y}$ path) can give, over some time interval, a strictly higher utility than the path equal to the average over time of the original path. So variability over time is, from the point of view of the allocation of consumption, desirable. But the opposite is true from the point of view of production: a variable consumption path can yield a strictly smaller asset accumulation than the path equal to the average over time. Making the variability occur in shorter and shorter time periods can reconcile the two desiderata. The utility is still going to be roughly equal to the average of utilities in different points in time. The loss in production is going to be smaller and smaller because the resulting oscillations in the x path are smaller and smaller. But the limiting measurable optimal path may not exist. The next example illustrates this in a simple economy.

An economy with no optimal growth path

The economy is described by $C = K = 1$, and $F(x, y) = f(x) - y$, with f strictly concave and such that $f'(1) = r$ and $f(1) = 1$. The initial condition for the capital stock is $x_0 = 1$.

The utility function U is monotonically increasing, but not concave. There is however a concave function V defined by $V(y) = U(y)$ for $y \in [0, 1/2] \cup [3/2, \infty]$, and $V(y) = \lambda U(1/2) + (1 - \lambda)U(3/2)$, for each $y = \lambda(1/2) + (1 - \lambda)(3/2)$, $\lambda \in (0, 1)$. Note that $V(y) > U(y)$, for $y \in (1/2, 3/2)$.

The optimal growth problem when the utility function is V (rather than U) has, given the choice of our initial condition, the unique solution $y(t) = x(t) = 1$, for all t. The value to the problem is $\frac{1}{2}(U(1/2) + U(3/2)) = V(1)$. Furthermore, since by construction $V(y) \geq U(y)$, for all y, the value to 1 with the utility function V is greater or equal to the value of the problem with the utility function U.

Consider now the growth problem with U. Fix an arbitrary time interval. Consider the consumption path $\hat{y}$ that alternates period by period the consumption levels 1/2 and 3/2. For r close to zero, the value of this path $U(\hat{y})$ is close to $\frac{1}{2}(U(1/2) + U(3/2)) = V(1)$. Since the production function is strictly concave, variability is costly. Thus, since the average (over time) consumption of the path is 1, $\hat{y}$ is, given initial condition, not feasible. However, by making the time period arbitrarily small, the cost of the variability converges to zero and the consumption path converges (in the appropriate topology, the narrow topology: see below, and [17] for details) to the time invariant path in the space of probability measures that assigns to both consumption levels, 1/2 and 3/2, a probability 1/2.

2.2 The relaxed problem

Let $\mathcal{M}_{+,1}(Y)$ denote the set of Borel probability measures on it. For any probability distribution $\beta \in \mathcal{M}_{+,1}(Y)$, and any real valued function $g(\cdot,\cdot)$, $g(\cdot,\beta)$ denotes the expectation of g according to β, i.e., $g(\cdot,\beta) = \int_Y g(\cdot,y)\beta(dy)$. Let $\hat{\beta} : \Re_+ \to \mathcal{M}_{+,1}(Y)$ be a measurable map. The relaxed (weak) problem associated to 1 is:

$$\sup_{\hat{\beta}} \int_0^\infty U(\hat{\beta}(t))dt \tag{3}$$

subject to the constraint:

$$\dot{x}(t) = F(\hat{x}(t), \hat{\beta}(t)), \hat{x}(0) = x_0, \hat{\beta}(t) \in M_{+,1}(Y) \tag{4}$$

In the programming problem (3), the planner can choose a path of lotteries over the consumption set and the feasibility constraints are written in average. The value of (3) is greater or equal to the value of (1). Furthermore, in (3), the utility function is linear in the control variable, $\beta \in \mathcal{M}_{+,1}(Y)$. Hence, under our specification, there exists an optimal solution $(\hat{x}, \hat{\beta})$ to (3), while (1) may not have an optimal solution. A first known result shows that the supremum of the problem (1) coincides with the value of the relaxed problem (3). Most importantly, a second known result shows that the sequence of deterministic trajectories with value approximating the supremum (1) converges in an appropriate topology (the narrow topology) to $(\hat{x}, \hat{\beta})$, the optimal solution to (3) (see ([17] for details).

These two results justify our characterization of efficient paths by means of the programming problem (3). However, the interpretation should be clear. The opportunity of selecting paths of lotteries over Y as well as the average form of the feasibility requirements are not special assumptions. They emerge from the optimal use of time as a convexifying device.

3 Intertemporal competitive equilibrium

The definition of competitive equilibrium is the natural extension of the concept of competitive equilibrium for an economy with complete markets where the household has a non-concave utility function. Faced with a budget constraint, the household will try to randomize consumption to achieve higher utility. If a complete market for lotteries over consumption sets exists, he will be able to do this. The randomization over consumption corresponds to the randomization he can achieve in the dynamic economy by altering consumption over time (*chattering*). The possibility of choosing lotteries over the space of basic consumption goods allows him to randomize at each point in time, rather than *across* time as he does with chattering. Hence, we identify the consumption set, at each t, with $\mathcal{M}_{+,1}(Y)$. The (lottery) price domain is $\mathcal{M}_{+,1}(Y)^*$, the dual of $\mathcal{M}_{+,1}(Y)$. It contains (an isomorphic copy of) the set of continuous functions on Y. We write $< \hat{p}, \hat{\beta} >$ the dual pair. We denote by L the Lebesgue measure on the real line. Firm and household allocations are vectors of paths

$$(\hat{x}_f, \hat{x}_h, \hat{a}_f, \hat{a}_h, \hat{\beta}_f, \hat{\beta}_h)$$

where, $\hat{x}_f(t)$ and $\hat{x}_h(t) \in \Re^K$ are, respectively, the supply and the demand of capital , $\hat{a}_f(t)$ and $\hat{a}_h(t) \in \Re^K$ are demand and supply of investment, and $\hat{\beta}_h(t)$ and $\hat{\beta}_f(t) \in \mathcal{M}_{+,1}(Y)$ are supply and demand of (lotteries over) consumption allocations. Prices are vectors of paths:

$$(\hat{q}, \hat{b}, \hat{p})$$

where $\hat{b}(t) \in \Re^K$ is the rental price of capital, $\hat{q}(t)$ is the price of investment good, and $\hat{p}(t) \in \mathcal{M}_{+,1}(Y)^*$ is the price of consumption allocations. The space of allocations is a subset of the space of measurable functions from $\Re_+$ to $\Re^{2K} \times \mathcal{M}_{+,1}(Y)$. The precise space is the set of measurable functions, $(\hat{x}, \hat{a}, \hat{\beta})$, such that the discounted L_1 norm is finite. Prices are linear functional on this space, hence measurable functions with finite, discounted essential sup norm. The inner product is defined in the natural way:

$$< (\hat{q}, \hat{b}, \hat{p})(\hat{x}, \hat{a}, \hat{\beta})) > \equiv \int_0^{+\infty} [\hat{q}(t)\hat{x}(t) + \hat{b}(t)\hat{a}(t) + < \hat{p}(t), \hat{\beta}(t) >]dt.$$

In the intertemporal economy, the firm and the household solve the following problems.

Definition 1 *For a given vector of prices, the firm's problem is:*

$$\max_{(x,a,\gamma)} \int_0^{+\infty} [-\hat{b}(t)\hat{x}(t) + < \hat{p}(t), \hat{\gamma}(t) > + \hat{q}(t)\hat{a}(t)]dt \tag{5}$$

subject to

$$\hat{a}(t) = F(\hat{x}(t), \hat{\gamma}(t)) \ L - almost\ every\ t; \tag{6}$$

Let $v(\hat{q}, \hat{b}, \hat{p})$ denote the value of the firm, i.e., $v(\hat{q}, \hat{b}, \hat{p}) = \int_0^{+\infty} [-\hat{b}(t)\hat{x}(t) + < \hat{p}(t), \hat{\gamma}(t) > + \hat{q}(t)\hat{a}(t)]dt$, where $(\hat{x}, \hat{a}, \hat{\beta})$ is a solution of the firm's problem (5) at $(\hat{q}, \hat{b}, \hat{p})$;

Definition 2 *For a given vector of prices and value of the firm $v(\cdot)$, the consumer's problem is:*

$$\max_{(x,a,\gamma)(t)} \int_0^{+\infty} e^{-rt} U(\hat{\gamma}(t))dt, \tag{7}$$

$$\textit{subject to} \int_0^{+\infty} < \hat{p}(t), \hat{\gamma}(t) > dt - \int_0^{+\infty} [\hat{b}(t)\hat{x}(t) - \hat{q}(t)\hat{a}(t)]dt \le v(\hat{q}, \hat{b}, \hat{p});$$

and

$$\hat{x}(0) = x_0,\ \dot{x} = \hat{a}(t),\ L - a.e.$$

Although U may be a non-concave function of $y \in Y$, U, and, therefore, $\int_0^{+\infty} e^{-rt} U(\cdot) dt$, is a linear function of the lotteries, $\gamma \in M_{1,+}(Y)$. Furthermore, if at equilibrium, preferences satisfy local non satiation, the path $(\hat{x}_h, \hat{a}_h)$ is chosen to maximize the wealth of the household. Wealth maximization is independent of the particular shape U and it is always a linear problem. Hence, 7 is a concave programming problem. The nature of the investment at time t, $\dot{x}(t)$ requires some clarification. Investment is a random variable, dependent on the realization of the random variable input. However, since this quantity enters linearly into the consumer's problem, we may assume that the consumer chooses a deterministic investment at each time. The market clearing condition on investment is satisfied L- almost surely because the market clearing condition on the stock of capital in each period is satisfied. If we write the market clearing condition explicitly, it is stated as an equality between the expected value of demand and supply of investment. We identify an economy with an array (U, F, x_0).

Definition 3 *A competitive equilibrium of the economy* (U, F, x_0) *is a vector of prices and allocations*

$$(\hat{q}, \hat{b}, \hat{p}; \hat{x}_f, \hat{x}_h, \hat{a}_f, \hat{a}_h, \hat{\beta}_f, \hat{\beta}_h)$$

such that

1. $(\hat{x}_f, \hat{x}_h, \hat{a}_f)$ *is a solution of the firm's problem (5);*
2. $(\hat{x}_h, \hat{a}_h, \hat{\beta}_h)$ *is a solution of the consumer's problem (7);*
3. *Markets clear:*

$$\hat{\beta}_f(t) = \hat{\beta}_h(t),\ \hat{x}_f(t) = \hat{x}_h(t),\ \hat{a}_f(t) = \hat{a}_h(t),\ L - \textit{almost every } t;$$

In the framework we have described, the economy is a neoclassical economy with concave (linear) preferences and convex technology. Hence under technical assumptions a competitive equilibrium exists, is efficient, and the second welfare theorem holds.

3.1 Static equilibrium.

The notion of static equilibrium is the restriction of the definition of intertemporal equilibrium to vectors of allocations and prices that are constant over time.

Definition 4 *A static equilibrium is a vector*

$$(q, a, p; x_f, a_f, \beta_f, \beta_h)$$

where $x \in R^K$, $\beta_f, \beta_h \in \mathcal{M}_{+,1}(Y)$*;* $p \in \mathcal{M}_{+,1}(Y)^*$, $q \in \Re^K$, *and* $v \in R$ *such that*

1. β_h *is a solution of the consumer's problem:*

$$\max_{\gamma \in \mathcal{M}_{+,1}(Y)} U(\gamma),\ \textit{subject to}\ <p, \gamma> \leq v; \tag{8}$$

2. *The firm's value gross of the capital expenses is*

$$v = <p, \beta_f> + qa'$$

3. (x_f, a_f, β_f) *is a solution of the firm's problem:*

$$\max_{(x', \beta)} -rqx' + [<p, \beta> + qa']\ \textit{subject to}\ a' \leq F(x', \beta) \tag{9}$$

4. *markets clear:*

$$F(x, \beta_f) = 0,\ \beta_h = \beta_f,\ a_f = 0$$

The relation between the static and the intertemporal equilibrium notions is fairly obvious. A stationary, intertemporal equilibrium is a static equilibrium. Viceversa, a static equilibrium (x, β, p, q) is a stationary equilibrium of the intertemporal economy, i.e., given the initial condition $x(0) = x$, the path, $(\hat{q}, \hat{b}, \hat{p}, \hat{x}, \hat{a}, \hat{\beta})(t) = (x, 0, \beta, e^{-rt}(q, 0, p))$, for all t, is an equilibrium of the intertemporal economy. Hence, the stationary allocation (x, β) is an efficient allocation of the intertemporal economy. A static allocation (x, β) is efficient if it is an efficient stationary allocation of the intertemporal economy, i.e., if there exists a $q \in \Re^K$ such that $(x, \beta) \in \arg\max U(\beta) + qF(x, \beta)$. (see, [13]).

4 Static and Intertemporal Sunspot Equilibria.

In agreement with the definitions of competitive equilibria, we give inter temporal and static definitions of sunspot equilibria. The common element is the existence of an exogenous (and, for sake of simplicity, time invariant) standard measurable space of extrinsic uncertainty (sunspots) $(\Omega, \mathcal{A})$ together with an exogenous probability, $\hat{\sigma}$. The definitions of sunspot equilibria is a simple restatement of the definitions of competitive equilibria in an environment where there is a complete set of contingent (on the realization $\omega \in \Omega$) commodities, but where lotteries are absent. Capital and investment good prices are restricted to be sunspot invariant. As already discussed, this is a consequence (given the sunspot invariance of the prices q) of the concavity in $(\hat{x}, \hat{a})$ of both the firm and the household programming problem. Let $\mathcal{B}_{\Re_+}$ denote the Borel sets of $\Re_+$. In the intertemporal economy, the space of allocations is the space of paths of $(\Omega \times \Re_+, \mathcal{A} \times \mathcal{B}_{\Re_+})-$measurable functions $(\hat{x}, \hat{a}, \hat{y})$: $\Omega \times \Re_+ \to \Re_+^{2K} \times Y$ satisfying the restriction "$(\hat{x}, \hat{a})(\omega, t)$ is $\omega-$invariant", with norm $\| (\hat{x}, \hat{y}) \|_{1,r} \equiv \int_0^{+\infty} e^{-rt} \| \hat{x}(t), \sup_{\omega \in \Omega} \hat{y}(\omega, t) \| dt$, finite. Prices are linear functional on this space, hence measurable functions with finite norm $\| (\hat{q}, \hat{b}, \hat{p}) \|_{\infty,r} \equiv \sup_{(t) \in R_+} e^{rt} \int_\Omega \| (\hat{q}, \hat{b}, \hat{p}(\omega))(t) \| \sigma(d\omega)$. Hence, $\hat{p}(\omega, t)$ is an element of $\Re_+^L$ for each (ω, t), and, thus, prices are linear (in contingent commodities). In the static economy, the space of allocations is the space of $(\Omega, \mathcal{A})-$measurable functions $(\hat{x}, \hat{a}, \hat{y}) : \Omega_+ \to \Re_+^{2K} \times Y$ satisfying the restriction that "$(\hat{x}, \hat{a})(\omega)$ is $\omega-$invariant. ", while prices are linear functionals over this space. As for the notion of competitive equilibrium, in a sunspot equilibrium, the equivalent constraint of equation 6 in the definition 1 reads

$$\hat{a}(t) = \int_\Omega F(\hat{x}(t), \hat{y}(\omega, t))\hat{\sigma}(d\omega) \; L - \text{almost every } t; \tag{10}$$

while for a static sunspot equilibrium, feasibility requires, in agreement with the definition of a static sunspot equilibrium:

$$0 = \int_\Omega F(x, \hat{y}(\omega)\hat{\sigma}(d\omega) \; L - \text{almost every } t; \tag{11}$$

We do not explicitly write down the definitions of static and intertemporal sunspot equilibria because, modulo the change in the allocation sets are identical to the definitions of intertemporal and static competitive equilibria.

When, judging the efficiency of sunspot allocations we use the following definition of efficiency:

Definition 5 *A feasible sunspot allocation $(\hat{x}, \hat{a}, \hat{y})$ of the intertemporal economy $(U, F, x_0, (\Omega, \mathcal{A}, \hat{\sigma}))$, is efficient if it does not exists a feasible (lottery) allocation $(\hat{x}', \hat{a}', \hat{\beta}')$ of the economy (U, F, x_0) such that*

$$\int e^{-rt}(\int_\Omega U(\hat{y}(\omega, t))\hat{\sigma}(d\omega))dt < \int e^{-rt}(U(\hat{\beta}'(t))dt.$$

5 Sunspot and Lottery Equilibrium Allocations

5.1 Sunspot Equilibria in Finite Economies

As already explained, two major differences between the concept of competitive equilibrium and sunspot equilibria are clear and crucial. The first difference concerns the nature of the consumption sets, while the second concerns the nature of prices. We start by analyzing the first difference.

5.1.1 Sunspot and Lottery Allocations

In economies with non convexities, there are two aspects to the link between lottery and sunspot allocations. 1) the space of sunspots $(\Omega, \mathcal{A}, \hat{\sigma})$ must be rich enough to provide the randomness necessary to reproduce lotteries. 2) Since $(\Omega, \mathcal{A}, \hat{\sigma})$ is the common randomization device, individual sunspot allocations must be appropriately correlated to be , $\hat{\sigma} - a.e.\omega$, feasible. This second aspect is obviously absent in our representative economy. Later, when we study the large economy, 2) will manifest necessarily itself. The first aspect is however present and it needs to be clarified.

In finite, complete markets economies different specifications of the sunspot space may yield different and Pareto ranked sunspot equilibrium allocations. In section 8, we work out an example of an economy where distinct specifications of the space of sunspot yield a countable infinity of Pareto ranked sunspot allocations. The economy is a static economy with a large number of individuals, but as we explain, it is as well a one period pure exchange economy. The existence of multiple sunspot equilibria provides the rationale for the search of ways to refine sunspot equilibria. ([7] ; [6]).

We identify the set of sunspots with the interval $[0, 1)$, their probability with the Lebesgue measure L and we allow for different specifications of the $\sigma-$algebra, $\hat{B}$, with $\hat{B} \subset B$, the Borel sets over the interval $[0, 1)$. As shown in [5], $([0, 1), B, L)$ (or any probability space isomorphic to it) accomplishes desideratum 1), in the following precise sense: Let C be a Borel subset of Y and observe that $\hat{y}^{-1}(C)$ is a Borel subset of $[0, 1)$, for $\hat{y} \in Y([0, 1), B)$, the set of $([0, 1], B) - measurable$ functions from $[0, 1)$ to Y. A lottery $\beta \in M_{1,+}(Y)$ and an allocation $\hat{y} \in Y([0, 1), B)$ are equivalent, $\hat{y} \sim \beta$, if for any C :

$$\beta(C) = L(\hat{y}^{-1}(C)) = L\{\omega : \hat{y}(\omega) \in C\}, \text{ and, therefore, } \beta = L \circ \hat{y}^{-1}.$$

[5], in Lemma 4, proves that for each $\beta \in M_{1,+}(Y)$, there are is $\hat{y} \in Y([0, 1), B)$ such that $\hat{y} \sim \beta$ and viceversa. Also, by the same argument, for each given sunspot space $(\Omega, \mathcal{A}, \hat{\sigma})$ and each $y \in Y(\Omega, \mathcal{A})$ there exists $\hat{y}^* \in Y([0, 1), B, L)$ such that $\hat{y}^* \sim \hat{y}$. Hence, $\{[0, 1), B, L\}$ is rich enough to reconstruct lotteries as well as any sunspot allocation measurable with respect to any arbitrarily given standard prob ability space $(\Omega, \mathcal{A}, \hat{\sigma})$. From now on, we identify a sunspot economy with the array $(U, F, x_0, \hat{B})$, $\hat{B} \subset B$. While, as already stated, a lottery economy is an array (U, F, x_0).

5.1.2 Constant price sunspot equilibria.

Again following [5], a key role in our analysis is played by a particular notion of sunspot equilibria, that we call constant prices sunspot equilibria, hereafter, CPSE. An inter temporal CPSE is an intertemporal sunspot equilibrium with $(\Omega, \mathcal{A}, \hat{\sigma}) = ([0, 1), B, L)$ and $\hat{p}(t, \omega) = p(t)$, for all $\omega \in [0, 1)$ and some $p(t) \in \Re^L$. The natural adaptation of the last definition to the static environment provides the definition of CPSE for the static economy. Whenever we talk about a CPSE of some economy, it is understood that $\hat{B} = B$ so that we can talk of a CPSE of the economy (U, F, x_0) without ambiguity.

In pure exchange economies with non-convexities, CPSE have a very powerful property. Under fairly general conditions, these two results hold true. 1) To each lottery equilibrium supported by linear prices it corresponds an equivalent CPSE and viceversa ([5], Theorem 3). 2) The sunspot invariance price restriction is without loss of generality.

Thus, in finite economies, whenever CPSE exist, they perform as well as linearly priced lotteries. However, the existence of CPSE (or, in general, of any sunspot equilibrium) is related to the form of the feasibility constraint. If the latter is a restriction on the average (sunspot or lottery) allocation (as in a large economy), CPSE exist, otherwise, their existence as well as the existence of any sunspot allocation may fail. Equivalently, as we will clarify later, CPSE exists if competitve (lottery) equilibria have linear supporting prices, otherwise they do not. The example that follows illustrates this simple point.

Example 6 *A one period economy without sunspot equilibria.*

We consider a one period, two commodities, pure exchange economy with a unique individual. The feasibility restricts the realizations of lotteries and sunspot allocations (as opposite to their average). In this economy, the unique efficient allocation can be supported in the lottery economy, but not in the sunspot one.

The consumer utility function is $U(y) = \sqrt[g]{(y_1)^2 + (y_2)^2}$, $g \geq 2$, and the endowments are $e = (1, 1)$. As we argue later, if $0 < g < 2$, the consumer problem in the sunspot economy does not have a solution. For all g, the unique efficient allocation coincides with e.

In the economy with lotteries, for given prices $p \in \mathcal{M}_{+,1}(Y)^*$, the consumer seeks a lottery $\beta \in \arg\{\max_{\beta \in M_{+,1}(Y)} U(\beta)$ subject to $\int_{y\in Y} p(y)\beta(dy) \leq p(e)\}$, while feasibility restricts lotteries into the set $\{\beta \in \mathcal{M}_{+,1}(Y) : \beta\{y \in Y : y \leq e)\}\} = 1$. The efficient lottery allocation is δ_e (i.e., the degenerate lottery that assigns probability 1 to e) supported by, for instance, the price $p(y) = U(y) \in \mathcal{M}_{+,1}(Y)^*$.

In the economy with sunspot space $([0, 1), B, L)$, we restrict, without loss of generality, prices to be $\omega - \mathit{invariant}$ and thus, for given $p \in \Re^2_{++}$ the individual seeks a sunspot allocation $\hat{y} \in \arg\{\max \int U(\hat{y}(\omega))d\omega$ subject to $\int p(\hat{y}(\omega) - e)d\omega \leq 0\}$. Feasibility restricts the sunspot allocations in the set $\{\hat{y} : L\{\hat{y} : \hat{y}(\omega) \leq e\} = 1$. The optimal solutions are, for each ω, the consumption bundle $(\frac{p_1+p_2}{p_1}, 0)$, if $p_1 < p_2$, $(0, \frac{p_1+p_2}{p_2})$, if $p_2 < p_1$, or the pair $(2, 0)$ and $(0, 2)$, if $p_1 = p_2$. Thus a CPSE (and more generally, any sunspot equilibrium) does not exist.

In the sunspot economy, for $g \in (0, 2)$, there is not a solution to the consumer problem. To make this point, suppose that $g = 1$ and $p_1 = p_2 = 1$. Pick $\varepsilon > 0$, set $\hat{y}(\sigma) = 0$, for $\sigma > \varepsilon$ and $\hat{y}(\sigma) = (2, 0)$, for $\sigma \in [0, \varepsilon)$. Then, $\int U(\hat{y}(\omega))d\omega = 4/\varepsilon$ which diverges to infinity as $\varepsilon \to 0$.

In the same economy, with a large number of identical individuals, sunspot equilibria exist and are efficient. We illustrate this point later on in section 8, where we work out an example of a static large economy that, as we explain, is as well a finite economy. However, the reason is simple. In the large lottery pure exchange economy, by the law of large numbers, feasibility restricts the average lottery consumption, i.e., $\int_Y (y - e)\beta(dy) \leq 0$. Apparently, the situation is different for the sunspot economy, where feasibility restricts, realization by realization, the average taken across individuals of the sunspot allocation. However, the sunspot allocation can be appropriately correlated across individuals in order to make the average taken across individuals equal to the average taken across sunspot realizations. ■

There are two important differences between the growth economy and the finite economy that change some aspects of the relation between lotteries and sunspot. First, in the growth economy, feasibility takes a particular form that never restricts allocations realization by realization. This is the feature that makes the existence of sunspot possible (for some specification of the technology) even for the representative individual growth model. Second, the use of time as an additional convexifying device combined with the sunspots result in the optimality of sunspot allocations. Contrary to what happens for finite economies, if sunspot equilibria exist they are efficient. To these topics we turn next.

5.2 Intertemporal Sunspot Equilibria.

As already said, in the growth economy, in addition to the probability space $([0,1),\hat{B},L)$, the representative household can use time as a convexifying device. The latter has two implications. First and in contrast to the finite economy results, intertemporal sunspot equilibria of the economy $(U,F,x_0,\hat{B})$ are efficient (in the sense of the definition (5) for any $\hat{B}\subset B$. Second, all intertemporal sunspot equilibria are, without loss of generality, CPSE. To avoid confusions, we are not showing that sunspot equilibria exits, but rather that, when they exists, they are efficient and CPSE. The first property implies that $([0,1),\hat{B})$ provides, at each t, enough variability to reproduce the optimal lottery and that the latter, by definition of sunspot, can be supported by linear prices. This second property has a key implication. When the representative agents maximize facing linear prices, they optimally select allocations yielding the value of the concave regularizations of their objective functions. The next Lemma makes precise this statement. Let $V=\ \text{co}\,U$ be the concave regularization of the utility function U, since Y is compact, V can be defined as:

$$V(y)\equiv \max_{\beta\in M_{1,+}(Y)}\{U(\beta),\text{ subject to } y(\beta)=y\},\text{ for } y\in Y.$$

For given $\hat{B}\subset B$, household wealth W and price trajectory $\hat{p}$ consider the two following programming problems:

$$(U)\qquad \max_{\hat{y}(t)\in Y([0,1),\hat{B})}\int_{\Re_+} e^{-rt}(\int_{[0,1)}(U(\hat{y}(\omega,t))d\omega)dt$$

subject to

$$\int_{\Re_+}(\int_{[0,1)}\hat{p}(\omega,t)\hat{y}(\omega,t)d\omega)dt\leq W$$

and

$$(V)\qquad \max_{\hat{y}(t)\in Y([0,1),\hat{B})}\int_{\Re_+} e^{-rt}(\int_{[0,1)}(V(\hat{y}(\omega,t))d\omega)dt$$

subject to

$$\int_{\Re_+}(\int_{[0,1)}\hat{p}(\omega,t)\hat{y}(\omega,t))d\omega)dt\leq W.$$

The next lemma explains the relations between the optimal solutions to (U) and (V). Its proof is deferred to the appendix.

Lemma 7 *Any optimal solution to (U), $\hat{y}^*$, is an optimal solution to (V). Furthermore,*

$$\int_{\Re_+} e^{-rt}(\int_{[0,1)} (U(\hat{y}^*(\omega,t))d\omega)dt = \int_{\Re_+} e^{-rt}(\int_{[0,1)} (V(\hat{y}^*(\omega,t))d\omega)dt.$$

We now exploit Lemma 7 to show that each intertemporal sunspot equilibrium of (U, F, x_0, B) is a sunspot equilibrium of (V, F, x_0, B). To simplify notation we denote by $\hat{\pi}$ prices and by $\hat{\xi}$ allocations.

Proposition 8 *Let $(\hat{\pi}^*, \hat{\xi}^*)$ be an intertemporal sunspot equilibrium of the economy $(U, F, x_0, \hat{B})$. Then, $(\hat{\pi}^*, \hat{\xi}^*)$ is a sunspot equilibrium of the economy $(V, F, x_0, \hat{B})$. Furthermore,*

$$\int_{\Re_+} e^{-rt}(\int_{[0,1)} (U(\hat{y}^*(\omega,t))d\omega)dt = \int_{\Re_+} e^{-rt}(\int_{[0,1)} (V(\hat{y}^*(\omega,t))d\omega)dt.$$

Proof The economies $(V, F, x_0; \hat{B})$ and $(U, F, x_0, \hat{B})$, when facing identical price trajectories, have identical profit and household's wealth maximization problems. Thus, we just need to prove that $\hat{y}^*$ is an optimal solution to

$$\max_{\hat{y}(t)\in Y([0,1),\hat{B})} \int_{\Re_+} e^{-rt}(\int_{[0,1)} (V(\hat{y}(\omega,t))d\omega)dt$$

subject to

$$\int_{\Re_+} (\int_{[0,1)} \hat{p}(\omega,t)\hat{y}(\omega,t))d\omega)dt \leq \int_{\Re_+} (\int_{[0,1)} \hat{p}(\omega,t)\hat{y}^*(\omega,t)d\omega)dt.$$

This follows immediately from Lemma 7 which implies as well the second part of the proposition. ∎

The economy $(V, F, r; \hat{B})$ is a concave economy. Therefore, sunspot do not matter. This is the key property that we exploit in order to show both that sunspot equilibria are efficient and that they are, without loss of generality, CPSE.

Proposition 9 *Let $(\hat{\pi}, \hat{\xi})$ be a sunspot equilibrium of the intertemporal economy $(U, F, x_0, \hat{B})$. Then: 1) $\hat{\xi}$ is an efficient allocation of the economies (U, F, x_0) and (V, F, x_0); 2) $(\hat{\pi}, \hat{\xi})$ is, without loss of generality, a CPSE of the economies (U, F, x_0) and (V, F, x_0).*

Proof By proposition 8, $(\hat{\pi}, \hat{\xi})$ is a sunspot equilibrium of $(V, F, x_0, \hat{B})$. Since the latter is a concave economy, by the First Fundamental Theorem of Welfare Economics, $\hat{\xi}$ is an efficient allocation (according to definition 5) of the economy (V, F, x_0). Suppose that, by contradiction, there exists a feasible lottery allocation $(\hat{x}^*, \hat{a}^*, \hat{\beta}^*)$ of the economy (U, F, x_0) such that:

$$\int_{\Re_+} e^{-rt} U(\hat{\beta}^*(t))dt > \int_{\Re_+} e^{-rt}(\int_\Omega (U(\hat{y}(\omega, t))d\omega)dt \tag{12}$$

Since, by the definition $V(y) \geq U(y)$, for all $y \in Y$, and since, by proposition 8, $\int_{\Re_+} e^{-rt}(\int_\Omega (U(\hat{y}(\omega,t))d\omega)dt = \int_{\Re_+} e^{-rt}(\int_\Omega (V(\hat{y}(\omega,t))d\omega)dt$, 12 implies that $\int_{\Re_+} e^{-rt} V(\beta^*(t))dt > \int_{\Re_+} e^{-rt}(\int_\Omega (V(\hat{y}(\omega,t))d\omega)dt$. A contradiction. Thus, 2) holds true. Since $\hat{\xi}$ is efficient for (V, F, x_0) and the latter is a concave economy, the following equalities hold true:

$$\int_{[0,1)} V(\hat{y}(\omega, t))d\omega = V(\int_{[0,1)} \hat{y}(\omega, t))d\omega), \text{ and} \tag{13}$$

$$\int_{[0,1)} F(\hat{x}(t), \hat{y}(\omega, t))d\omega = F(\int_{[0,1)}(\hat{x}(t), \hat{y}(\omega, t))d\omega)).$$

Thus, by 13, $(\hat{x}, \hat{a}, \bar{y})$ is an efficient allocation of the intertemporal economy (V, F, x_0), for $\bar{y}(t) = \int_{[0,1)} \hat{y}(\omega, t)d\omega$. Therefore, there exists a sunspot invariant price $(\hat{q}^*, \hat{b}^*, \hat{p}^*)$ that supports $(\hat{x}, \hat{a}, \bar{y})$ as a (degenerate) sunspot equilibrium of the economy (V, F, x_0, B^*), for any $B^* \subset B$. However, again by 13, $(\hat{q}^*, \hat{b}^*, \hat{p}^*)$ supports also the allocation $(\hat{x}, \hat{a}, \hat{y})$ as an intertemporal sunspot equilibrium of the economy (V, F, x_0, B^*), for any $\hat{B} \subset B^* \subset B$. Hence, to conclude the argument we need to show that $(\hat{q}^*, \hat{b}^*, \hat{p}^*)$ supports $(\hat{x}, \hat{a}, \hat{y})$ as an intertemporal sunspot equilibrium of the economy (U, F, x_0, B^*), for any $\hat{B} \subset B^* \subset B$.

Since the economies $(V, F, r; B^*)$ and $(U, F, r; B^*)$, when facing identical price trajectories, have identical profit and household's wealth maximization problems, it suffices to show that $\hat{y}(t)_{t \geq 0}$ is an optimal solution to

$$\max_{y(t) \in Y([0,1), \hat{B})} \int_{\Re_+} e^{-rt}(\int_{[0,1)} (U(\hat{y}(\omega, t))d\omega))dt$$
$$\textit{subject to} \int_{\Re_+} \hat{p}^*(t)(\int_\Omega \hat{y}(\omega, t))d\omega)dt \leq \int_{\Re_+} \hat{p}^*(t)(\int_{[0,1)} \hat{y}(\omega, t))d\omega)dt$$

Suppose by contradiction that there exists a budget feasible allocation $\hat{y}'$ such that $\hat{y}'(t) \in Y([0,1), B^*)$, for all t, and

$$\int_{\Re_+} e^{-rt}(\int_{[0,1)} (U(\hat{y}'(\omega, t))d\omega)dt > \int_{\Re_+} e^{-rt}(\int_{[0,1)} (U(\hat{y}(\omega, t))d\omega))dt$$

$$= \int_{\Re_+} e^{-rt}(\int_{[0,1)} (V(\hat{y}(\omega, t))d\omega)dt.$$

However, the definition of V and the previous inequality imply that

$$\int_{\Re_+} e^{-rt}(\int_{[0,1)} (V(\hat{y}'(\omega, t))d\omega)dt > \int_{\Re_+} e^{-rt}(\int_{[0,1)} (V(\hat{y}(\omega, t))d\omega)dt.$$

A contradiction. ∎

The converse of the last proposition is not true. We will show that there are competitive equilibria that can never be implemented as sunspot equilibria. Thus, we look for economically meaningful restricted environments, where the equivalence of sunspot and competitive equilibria is guaranteed. One restriction that delivers the equivalence result is on the technology F, we call it "semi-linearity". The second important environment is the large economy. Next we turn to semi-linear economies.

5.3 Sunspot equilibria and semi linear economies

An economy (U, F, r) is semi-linear if $F(x, y) = f(x) + By$ for some (concave) function $f : R^K_+ \to R^K$ and matrix B of dimension $L \times K$. Semi-linear economies have an appealing feature: equilibrium prices are (without loss of generality) linear, i.e., $< p, \beta >= p \int y\beta(dy)$, for some $p \in R^L_+$. The latter is obviously a necessary feature to allow for the existence of sunspot equilibria. First we show that prices are indeed linear in a semi-linear economy and than we exploit this feature to construct equivalent CPSE. The next proposition shows that lottery equilibrium allocations of a semi-linear economy are closely related to the equilibrium allocations of the concavified version of the economy. Most importantly, the supporting prices are identical and, therefore, linear. Let $(\hat{a}^*, \hat{x}^*, \hat{y}^*)$ be a feasible allocation of the semi-linear economy (V, F, x_0) and let $(\hat{a}, \hat{x}, \hat{\beta})$ be an allocation of the semi-linear economy (U, F, x_0). The two allocations are equivalent if $(\hat{a}^*, \hat{x}^*)(t) = (\hat{a}, \hat{x})(t)$, $U(\hat{\beta}(t)) = V(\hat{y}^*(t))$ and $y(\hat{\beta}(t)) = \hat{y}^*(t)$, for $L - a.e.t.$

Proposition 10 *Let $F(\cdot) = f(\cdot) + By$. The economies (U, F, r) and (V, F, r) have equivalent efficient allocations. Furthermore, without loss of generality, the supporting prices are in both economies equal and linear.*

Proof Let $(\hat{a}, \hat{x}, \hat{y})$ be an efficient allocation of the semi-linear economy defined by (V, F, x_0). Let $\hat{\beta}(t) \in \arg\{\max_{\gamma \in M_{1,+}} U(\gamma)$ subject to $y(\gamma) = \hat{y}(t)\}$. By the definition of least concave, $V(\hat{y}(t)) = U(\hat{\beta}(t))$ and by the semi-linearity of the technology, $(\hat{a}, \hat{x}, \hat{\beta})$ is feasible. Thus, the allocations $(\hat{a}, \hat{x}, \hat{y})$ and $(\hat{a}^*, \hat{x}^*, \hat{\beta})$ are equivalent. Since, $(\hat{a}, \hat{x}, \hat{y})$ is efficient for (V, F, x_0), the definition of V implies that $(\hat{a}, \hat{x}, \hat{\beta})$ is efficient for (U, F, x_0). Conversely, let $(\hat{a}, \hat{x}, \hat{\beta})$ be an allocation of the semi-linear economy (U, F, x_0). Then, by the semi-linearity of the technology, $\hat{\beta}(t) \in \arg\{\max U(\gamma)$ subject to $y(\gamma) = y(\hat{\beta}(t))\}$. Thus, $U(\hat{\beta}(t)) = V(y(\hat{\beta}(t))$. Then, the allocations $(\hat{a}, \hat{x}, \hat{\beta})$ and $(\hat{a}, \hat{x}, \hat{y}_{\hat{\beta}})$, $\hat{y}_{\hat{\beta}}(t) = y(\hat{\beta}(t))$, for all t, are equivalent and, quite obviously, $(\hat{a}, \hat{x}, \hat{y}_{\hat{\beta}})$ is an efficient allocation of (V, F, x_0). Thus the first part of the proposition holds true. Consider a pair of efficient and equivalent allocations $(\hat{a}, \hat{x}, \hat{\beta})$ and $(\hat{a}, \hat{x}, \hat{y})$. By the concavity of (V, F, x_0), there exists a price $(\hat{q}, \hat{b}, \hat{p})$, with $\hat{p}(t) \in R^L$, for all t, supporting $(\hat{a}, \hat{x}, \hat{y})$ as a competitive equilibrium allocation. Thus, we need to show that $(\hat{q}, \hat{b}, \hat{p})$ supports $(\hat{a}, \hat{x}, \hat{\beta})$ as an equilibrium allocation of the economy (U, F, x_0). Since the firm profit and the household wealth maximization problems are identical in the two economies, we just have to show that $\hat{\beta}$ is an optimal solution to: $\max \int_{\Re_+} e^{-rt}U(\hat{\gamma}(t)dt$ subject to $\int_{\Re_+} \hat{p}(t)y(\hat{\gamma}(t))dt \leq \int_{\Re_+} \hat{p}(t)\hat{y}(t))dt$ Once again, the definition of V implies the claim. ∎

The proposition allows for an immediate characterization of intertemporal equilibria of the semi-linear economies in terms of CPSE sunspot equilibria.

Corollary 11 *Let $(\hat{q}, \hat{b}, \hat{p}, \hat{x}, \hat{a}, \hat{\beta})$ be a competitive equilibrium with linear prices of the semi-linear economy (U, F, r). Then, there exists a consumption sunspot allocation $\hat{y}$, with $\hat{y}(t) \sim \hat{\beta}(t)$, for all t, such that the vector $(\hat{q}, \hat{b}, \hat{p}, \hat{x}, \hat{a}, \hat{y})$ is an intertemporal CPSE.*

Proof This is an immediate consequence of the equivalence relation $\sim$. It suffices to observe that, for any given measurable function g and any allocation $\hat{y} \in Y([0,1), B)$, by the standard change of variable formula, $\int_{[0,1)} g(\hat{y}(\omega))\lambda(d\omega) = \int_Y g(y)L \circ \hat{y}^{-1}(dy)$. However, for $\hat{y} \sim \beta$, $\beta = L \circ \hat{y}^{-1}$, hence $\int_Y g(y)\lambda \circ y^{-1}(dy) = \int_Y g(y)\beta(dy)$. It follows that at common prices $(\hat{q}, \hat{b}, \hat{p})$, equivalent paths $(\hat{x}, \hat{a}, \hat{\beta})$ and $(\hat{x}, \hat{a}, \hat{y})$ yield the same value of profits, overall utility and budget constraints. Since for each $\beta \in M_{1,+}(Y)$ there exists equivalent $\hat{y} \in Y([0,1), B)$, and viceversa, the latter implies the claim. ∎

Proposition 10 implies that every competitive equilibrium can be supported by linear prices. Thus, Corollary 11 implies that every competitive equilibrium allocation can be supported as a sunspot equilibrium allocation. Furthermore, Proposition 9, imply that also of the converse of the claim in Corollary 11 holds true. Hence, lotteries and sunspot are in the semi-linear environment equivalent. Propositions 9 has another interesting implication. Consider $(\hat{x}, \hat{a}, \hat{y})$ an inter-temporal sunspot equilibrium allocation of the economy $(U, F, x_0, \hat{B})$, with F (not necessarily a semi-linear). By Proposition 9 , $(\hat{x}, \hat{a}, \hat{y})$ is an inter temporal sunspot equilibrium allocation of $(V, F, x_0, \hat{B})$. Hence, $\int_{[0,1)} V(\hat{y}(\omega, t)d\omega = V(\int_{[0,1)} \hat{y}(\omega, t)d\omega$ and $\int_{\{0,1)} F(\hat{x}, \hat{y}(\omega))d\omega = F(\hat{x}, \int_{[0,1)} \hat{y}(\omega)d\omega$, for $L - a.e.t.$ Thus, both V and F are linear in the in the relevant range. In other words, if there are intertemporal sunspot equilibria, the representative agent economy is without essential loss of generality, semi-linear.

6 Non Existence of Sunspot Equilibria.

The converse of proposition 9 (point 1) is not true. There are intertemporal competitive equilibria of economies (U, F, x_0) that do not allow for an equivalent CPSE. Equivalently, by proposition 9 (point 2), there are intertemporal competitive equilibrium allocations of (U, F, x_0) that cannot be supported as intertemporal sunspot equilibria. There are also economies (U, F, x_0) that do not have intertemporal sunspot equilibria at all. In this section, we construct examples by using the static notion of equilibrium. However, before doing that we need to clarify the relation between static and intertemporal sunspot equilibria.

6.1 Static and Stationary Sunspot Equilibria

The relation between the static and the intertemporal notion of lottery allocations is, as already mentioned, clear. Efficient static allocations are efficient, stationary intertemporal allocations. Also, static competitive equilibria are stationary, intertemporal competitive equilibria. However, there might be static sunspot equilibria that are not intertemporal (stationary) sunspot equilibria. The reason is simple. In the static formulation of the model time is absent, while in the intertemporal version, agents can use time to generate lotteries over basic commodities. When prices are, as in the sunspot case, linear, the agents can reproduce lotteries that yield the least concave utility function. Hence, only static sunspot equilibrium allocations yielding the least concave utility can survive the test of time, i.e., they are stationary, intertemporal sunspot equilibria. This is summarized in the next proposition.

Proposition 12 *A static sunspot equilibrium of the economy $(U, F, \hat{B})$ is a stationary, intertemporal sunspot equilibrium only if it is a static sunspot equilibrium of the "concavified "economy $(V, F, r; \hat{B})$ (as well as, a stationary, intertemporal sunspot equilibrium of the concavified economy $(V, F, r; \hat{B})$).*

Proof The argument is a trivial consequence of proposition 9. ■

6.2 Examples

We construct (open sets of) static economies (U, F) that have:

1) a continuum of Pareto ranked static sunspot equilibria associated to distinct specifications of the sunspot space;

2) efficient allocations supportable by static sunspot equilibria, but not by CPSE, and an empty set of intertemporal sunspot equilibria for some value of the initial capital stock x_0 and for all $\hat{B} \subset B$;

3) with an empty set of static sunspot of equilibria for all $\hat{B} \subset B$.

By proposition 9, the Pareto ranked static sunspot allocation cannot be intertemporal sunspot allocations and intertemporal sunspot equilibria are, without loss of generality, CPSE. Thus, point 1) does not apply to the intertemporal economy, while 2) implies that sunspot and lottery intertemporal equilibrium allocations are not equivalent and that the set of intertemporal sunspot equilibria may be empty. Thus, 1) - 3) point out that, when preferences are not convex, static sunspot equilibria are a inadequate tool to analyze stationary equilibrium allocations of growth economies.

Proposition 13 *Static sunspot equilibria may fail to exist. When they exist they may be indeterminate and inefficient. Efficient stationary allocations may be decentralized as static sunspot equilibria, but not as static CPSE.*

Example 14 *An economy may have a continuum of static sunspot equilibria, all Pareto ranked.*

The economy is described by $C = K = 1$, and $F(x, y) = f(x) - y^\theta$, $\theta \geq 1$, with f strictly concave and such that $f'(1) = r$. The utility function has $U'(0) = +\infty$, is monotonically increasing in an interval $[0, M]$, and is such that co $U(1) > U(1)$. For example the following utility function satisfies all these conditions:

$$U(y) = \min\{\max\{y^\alpha, y\}, 2M - y\}, \tag{14}$$

where $\alpha \in (0, 1)$ and $M > 2$, arbitrarily large.

We restrict the space of extrinsic uncertainty to two points $\omega \in \{1, 2\}$ with $\sigma(1) = \sigma$. We set $q = 1/\theta$. The first order conditions for the firm's problem (recall that this problem is concave) give that the equilibrium capital is

$$x^* = 1 \tag{15}$$

and the firm's supply of y is

$$\hat{y}_f(\omega) = (\hat{p}(\omega))^{\frac{1}{\theta-1}}, \ \omega = 1, 2 \tag{16}$$

when $\theta > 1$. When $\theta = 1$, the equilibrium prices are $\hat{p}(\omega) = 1$, $\omega = 1, 2$. At the equilibrium prices, the set of optimal y's is the set of non negative quantities.

For given $\theta \geq 1$, a static, sunspot equilibrium allocation $\hat{y}^* = (\hat{y}^*(1), \hat{y}^*(2))$ satisfies:

1. the solution of the consumer's problem,

   $$\max_y \sum_\omega \sigma(\omega)U(\hat{y}(\omega)), \text{ subject to } \sum_\omega \sigma(\omega)(\hat{y}^*(\omega))^{\theta-1}\hat{y}(\omega) \leq \sum_\omega \sigma(\omega)(\hat{y}^*(\omega))^\theta$$

   subject to coincides with $\hat{y}^*$,

2. the market clear: $\sum_\omega \sigma(\omega)(\hat{y}^*(\omega))^\theta = 1$.

When $\max\{\frac{1}{\sigma\hat{p}(1)}, \frac{1}{(1-\sigma)\hat{p}(2)}\} < M$ or equivalently, $\frac{1}{M\hat{p}(1)} < \sigma < \frac{\hat{p}(2)M-1}{(\hat{p}(2)M)}$, the utility function is monotonically increasing in the budget feasible set, so the solution of the consumer problem will satisfy the budget constraint as an equality. The first order condition that marginal utilities are *equal* at the two choices $\hat{y}(\omega)$ is necessary. Consider for example the utility function in equation (14). Note that the equilibria are symmetric: to any equilibrium at $(\sigma, 1 - \sigma))$ corresponds an equilibrium at $(1 - \sigma, \sigma)$, so we only consider $\sigma \equiv \sigma(1) \leq 1/2$. The Inada condition that $U'(0) = +\infty$ insures that $y^* \gg 0$. Let, without loss of generality, $\alpha\hat{p}(2) < \hat{p}(1)$. (Otherwise relabel the states) Since, $\lim_{y\uparrow 1} U'(y)/\hat{p}(1) = \alpha/\hat{p}(1) < \lim_{y\downarrow 1} U'(y)/\hat{p}(2) = 1/\hat{p}(1)$, the budget feasible allocation $\hat{y}(\omega) = 1$, $\omega = 1, 2$ is not optimal. Then, the sunspot equilibrium allocation $\hat{y}^*$ together with the supporting prices $\hat{p}^*$ is a solution to the following system of equations:

$$\hat{y}(1) = (\frac{\alpha\hat{p}(2)}{\hat{p}(1)})^{1-\alpha}, \ \sum_\omega \sigma(\omega)\hat{p}(\omega)\hat{y}(\omega) = 1, \text{ and } \hat{p}(\omega) = (\hat{y}(\omega))^{\theta-1}, \ \omega = 1, 2. \tag{17}$$

The latter is a system of 4 equations in 5 unknowns $(\hat{y}, \hat{p}, \sigma)$. The solution changes with σ and this gives the continuum of equilibria. The value depends on σ, and the equilibria are Pareto ranked. The case $\theta = 1$ is simple:

1. there is an equilibrium for any $\sigma \in (1/M, 1/2]$;
2. the allocations are different for different σ:
$$\hat{y}^*(\sigma) = (\frac{1-(1-\sigma)\alpha^{\frac{1}{1-\alpha}}}{\sigma}, \alpha^{\frac{1}{1-\alpha}}); \tag{18}$$
3. the value is strictly decreasing in σ, and given by:
$$1 + (1-\sigma)(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}). \tag{19}$$

The allocations described in 2. satisfy $\hat{y}^*(2) \in (0,1)$ and $y^*(1) \in (1, M)$, for $\sigma \in (1/M, 1/2]$. The matrix of partial derivatives of the system of equations 17 with respect to (y, p) is for, $\theta = 1$ and for any $\sigma \in (1/M, 1/2]$, invertible. Hence, for $\sigma' \in (1/M, 1/2]$, by the implicit function theorem, for each (θ, σ) in a neighborhood of $\theta = 1$ σ', there exists a sunspot equilibrium. ■

The economy in the next example, has a unique and stationary efficient allocation that can be supported as a static sunspot equilibrium, but not as a static CPSE. Thus, by proposition 9, intertemporal sunspot equilibria are CPSE. Thus, the example shows the existence of economies with an empty set of intertemporal sunspot equilibria (but with a non-empty set of static sunspot equilibria). The reason is simple: there are static economies where the efficient lottery cannot be supported by linear prices and, hence, in virtue of the isomorphism theorem, by a CPSE. Quite surprising, they can be supported by sunspot equilibria that display sunspot price variability. Furthermore, the example is in contrast with the findings of [5] for finite, pure exchange economies. The existence of sunspot equilibria which are (genuinely) not CPSE is, in that context, possible only with finite sunspot state space and discrete consumption sets.

Example 15 *There are economies (U, F) with efficient allocations supported as static sunspot equilibria, but not as static CPSE. These economies have empty set of intertemporal sunspot equilibria.*

The economy is defined by the technology of the previous example and the utility function in equation (14), with parameter values (α, θ), $\theta > 1$ and $\alpha \in (0, 1)$. The optimal amount of capital is equal to 1, for all θ. The optimal lottery β^* is found by solving the following steps:

1) for given $q > 0$, find the set $Y^*(q) = \arg\max_y U(y) - qy^\theta$

2) find a price for the investment good q and a probability distribution over $Y^*(q)$ such that $y^\theta(\beta) = 1$ for $\beta \in M_{+,\cdot}(Y^*(q))$. Recall that, by definition, for y and $y' \in Y^*(q)$, $U(y') - q(y')^\theta = U(y) - q(y)^\theta$.

Since $\lim_{y\uparrow 1} U'(y) = \alpha < \lim_{y\downarrow 1} U'(y) = 1$, the deterministic allocation $\beta^* = \delta_1$ is suboptimal for all parameter values, $\alpha \in (0, 1)$ and $\theta > 1$.

To characterize the optimal lottery we seek values of the four unknowns (y_1, y_2, β, q) that satisfy four equations (market clearing, two optimality conditions, $y_i \in Y(q)$, and $U(y_1) - q(y_1)^\theta = U(y_2) - q(y_2)^\theta$). In order to avoid complications, we restrict the parameters values so that the solution satisfies $y_2^* < M$. Thus, we constraint (α, θ) in the set $P(M) = \{(\alpha, \theta) : \alpha \in (0, 1), \theta > 1 \text{ and } y_2^* < M\}$, an open subset of R_+^2. By straightforward computations, the unique solution is:

$$y_1^* = (\tfrac{\theta-\alpha}{\theta-1})^{\frac{\theta-1}{\theta(1-\alpha)}}(\alpha)^{\frac{1}{\theta(1-\alpha)}},\ y_2^* = (\tfrac{\theta-\alpha}{\theta-1})^{\frac{\theta-\alpha}{\theta(1-\alpha)}}(\alpha)^{\frac{\alpha}{\theta(1-\alpha)}},$$
$$q^* = \theta^{-1}(\tfrac{\theta-1}{\theta-\alpha})^{\frac{(\theta-\alpha)(\theta-1)}{\theta(1-\alpha)}}(\alpha)^{\frac{-\alpha(\theta-1)}{\theta(1-\alpha)}},\ \text{and } \beta^* = \frac{{y_2^*}^\theta - 1}{{y_2^*}^\theta - {y_1^*}^\theta}.$$

For given $\theta > 1$, both y_1^* and y_2^* are strictly increasing functions of α. For $\alpha = 0$, $y_2^* = \frac{\theta}{\theta-1} > 1$ and $y_1^* = 0$, while for $\alpha = 1$, $y_1^* = 1$ (and $y_2^* = \infty$). Thus, for every $\alpha \in (0, 1)$ and $\theta > 1$, the optimal lottery has a two point support with $0 < y_1^* < 1 < y_2^* < M$. It is easy, but tedious to show that, $\hat{a}(t) = 0$, $\hat{x}(t) = 1$ and $\hat{\beta}(t) = \beta^*$ is the unique efficient allocation of our growth economy for $x_0 = 1$.

In order to show that the efficient stationary allocation can be supported as a static sunspot equilibrium, we take $\Omega = \{1, 2\}$ with $\sigma = \beta^*$. Profit maximization pins down the supporting prices as:

$$\hat{p}^*(\omega) = q^*\theta(y_\omega^*)^{\theta-1},\ \omega = 1, 2.$$

$\frac{\hat{p}^*(1)}{\hat{p}^*(2)}$ is equal to the value of the marginal rate of substitution, $\frac{\partial U(y_1^*)/\partial y}{\partial U(y_2^*)/\partial y}$ at the efficient allocation. This is not surprising, since both y_1^* and y_2^* belong to regions where the utility function is concave. Furthermore, for each given $M > 2$, $\frac{\hat{p}^*(1)}{\hat{p}^*(2)} \neq 1$, for an open and dense set of parameter values in $P(M)$. Hence, for (α, θ) in an open and dense subset of $P(M)$, efficient allocation are supported as sunspot equilibria, but not as CPSE. Thus, by proposition 9, the uniqueness of the efficient allocation implies that, for these parameter values, the set of sunspot equilibria is empty. ∎

The last example has shown that the set of intertemporal sunspot equilibria may be empty, the next shows that even the set of static sunspot equilibria may be empty. Since static equilibria have a somewhat minor interest, we skip details.

Example 16 *An economy (U, F) does not have static sunspot for any σ-algebra $\hat{B} \subset B$.*

Let $C = K = 2$, and

$$F_1(x_1, y_1) = \frac{1}{2}(f_1(x_1) - y_1^2),\ \text{and } F_2(x_2, y_2) = f_2(x_2) - y_2 \tag{20}$$

while the preferences are defined by

$$U(y_1, y_2) = y_2 + u(y_1), \tag{21}$$

where for $\alpha < 1$ and $\epsilon > 0$ and small enough, u is defined by

$$u(y_1) = \begin{array}{c} (\frac{11}{1-\epsilon})y_1 - (10+\epsilon), \text{ for } y_1 \in [0, 1-\epsilon), \\ y_1 + (\alpha/2)[(y_1-1)^2 - (\epsilon)^2], \text{ for } y_1 \in [1-\epsilon, 1+\epsilon] \\ y_1, \text{ for } y_1 > 1+\epsilon. \end{array}$$

The function u is continuous and piecewise differentiable, and therefore so is U; for ϵ small enough, U is strictly increasing.

Since the functions F_i are, for $i = 1, 2$, separable in x and y, the optimal capital stocks are independent of prices and are found by solving:

$$\max_{x_i} a_i f(x_i) - r x_i,$$

for $i = 1, 2$ and $a_1 = 1/2$ and $a_2 = 1$. Since f_i are strictly concave the optimizers x_i^* are unique and we take them to satisfy:

$$f(x_i^*) = x_i^* = 1, \text{ for } i = 1, 2.$$

In this static economy, although the utility function is not concave, the planner allocation, (x^*, y^*), is deterministic, it is equal to $(1, 1, 1, 1)$. However, $(1, 1)$ is a non-concave region of the utility function. This is the driving force of the next proposition.

Proposition 17 *There are no sunspot equilibria of the static economy defined by (20), (21).*

Proof See [13]. ∎

7 The Large Economy

In this section we study the version of the economy with a large number of identical households. For large numbers of individuals we mean (as usual) that the law of large numbers holds true. Namely, if we assign to each household an *i.i.d.* lottery $\beta \in M_{1,+}(Y)$, the pro capita average allocation is equal (with probability one) to $y(\beta) = \int y\beta(dy)$. The particular specification the measurable space of households is not important, provided that the law of large numbers holds. We take it to be $\{[0, 1), B, L\}$ (with the usual caveat that applies to the validity of the Law of Large numbers with a continuum of random variables). As for any large economy, we assume that feasibility is a restriction over the random realizations of the pro capita total output.

We limit attention to efficient allocations yielding the same level of utility to each household, "equal utility allocations." This is a natural restriction dictated by our framework. In the market economies, households face the same prices and they have identical wealth, choice sets and preferences. Hence, at any conceivable competitive equilibrium, households get identical utility levels. The characterization of efficient "equal utility allocations" is substantially simplified by the existence of a large number of households. The set of consumption allocations, at each instant, can be identified just with the set $M_{1,+}(Y)$. To each $\beta \in M_{1,+}(Y)$ it is associated the pro capita consumption, $y(\beta)$. Hence, feasibility is just a restriction over average output. The reason is obvious. First assign to each household in the economy independent lotteries identically equal to β. Then invoke the law of large number and conclude that the average output is equal (with probability 1) to $y(\beta)$. One can show that by considering more general random allocations, efficiency is not enhanced, [14].

It should be clear at this point that a large economy has basically the same properties of a semi-linear economy. Every time feasibility is a restriction on the average values of the lotteries, the basic properties of equilibrium and efficient path of the economy (U, F, r) are closely captured by those of the concavified economy (V, F, r). This is the substance of the arguments, which, are obviously very similar to the ones used for the semi-linear case.

7.1 Equal Utility Efficient Paths

Equal utility efficient paths are the solution to the following programming problem:

$$\max_{(\hat{x},\hat{\beta})} \int e^{-rt} U(\hat{\beta}(t))dt \text{ subject to } F(\hat{x}(t), y(\hat{\beta}(t))) \geq \dot{x}(t), \hat{x}(0) = x_0. \qquad (22)$$

The optimal solutions to (22) can be characterized in a simple way by making reference to the "concavified" economy (V, F, r) Hence, consider the following programming problem, where, given the concavity of V, the planner optimizes within deterministic allocations:

$$\max_{(\hat{x},\hat{y})} \int e^{-rt} V(\hat{y}(t))dt \text{ subject to } F(\hat{x}, \hat{y}(t)) \geq \dot{x}(t), \hat{x}(0) = x_0. \qquad (23)$$

Let $V(x_0)$ be the value of (23) and $U(x_0)$ be the value to (22). Then:

Proposition 18 $(\hat{x}, \hat{y})$ *is an optimal solution to (22) if and only if* $(\hat{x}, \hat{\beta})$ *with* $U(\hat{\beta}(t)) = V(\hat{y}(t))$ *and* $y(\hat{\beta}(t)) = y^*(t)$*, for all* $L - a.e.t$*, is an optimal solution to (23). Hence,* $V(x_0) = U(x_0)$*.*

Proof The proof is an easy consequence of the definition of least concave function. ■

7.2 Lottery and sunspot equilibria

The large competitive economy with a complete set of markets for lotteries is identical to the one we have already described. The only difference is in the presence of a large number of households. In order to characterize the equilibria of our large economy, we follow the same technique adopted for the characterization of efficient paths. We study the "concavified" economy where, rather than U, V is the utility function of the households.

The efficient allocations of the concavified economy and the efficient allocations of the large economy are equivalent in the sense of proposition 18. Furthermore, for both economies, competitive allocations and efficient allocations coincide. Hence, the two economies have equivalent competitive allocations. In the next proposition, we state that they have identical supporting prices. Therefore, supporting prices of large economies are linear in commodities. This is the key fact for sunspot decentralization of the efficient paths.

Proposition 19 *Let $(\hat{x}, \hat{a}, \hat{y})$ be an efficient path of the "concavified" economy supported by prices $(\hat{q}, \hat{b}, \hat{p})$. Then the efficient path $(\hat{x}, \hat{a}, \hat{\beta})$, with $U(\hat{\beta}(t)) = V(\hat{y})$ and $y(\hat{\beta}(t)) = \hat{y}(t)$, of the large economy is supported by the same prices.*

Proof The proof is basically identical to the proof of proposition 10 and it is, therefore, omitted. ∎

7.2.1 Constant Prices Sunspot equilibria

Propositions 10, 18 and 19 establish a strong similitude between representative economies with semi-linear technologies and economies with a large number of households. Since feasibility is a restriction on average consumptions allocations, both economies display trajectories "similar" (in the precise sense stated in the various propositions) to the trajectories of their corresponding "concavified" economy. Most importantly, both economies have equilibrium prices linear in commodities. It is not a surprise, therefore, that the results of Corollary 11 generalize to large economies. However, there is a difference between this two environments that makes the extension non immediate. In the representative household semi-linear environment, aggregate and individual consumption are obviously equal. Consumption may be random, although only its average value is of importance for feasibility. In the large economy, individual and pro capita consumption are very different. The first may display randomness, while the second is deterministic and equal to the average value (over lottery realizations) of the random individual consumptions. Hence, as already mentioned, sunspot allocation have in the large economy the additional task of creating enough correlation among individual allocations in order to make the pro capita consumption sunspot independent and feasible. A sunspot consumption allocation of the large economy is a measurable map $\hat{A} : [0, 1)^2 \times R_+ \to Y$, where $\hat{A}(h, \omega, t)$ denotes the consumption bundle consumed by individual h at t when ω realizes. Given an initial condition x_0, *an* intertemporal $CPSE$ of the large economy is an $\omega - invariant$ trajectory of prices $(\hat{q}, \hat{b}, \hat{p})$, of investments, capital goods and pro capita consumption allocations $(\hat{x}, \hat{a}, \hat{y})$ and a consumption allocation $\hat{A}$ such that *i)* $(\hat{x}, \hat{a}, \hat{y})$ is profit maximizing at $(\hat{q}, \hat{b}, \hat{p})$; *ii)* $(\hat{x}, \hat{a}, \hat{A}(h))$ is utility maximizing at $(\hat{q}, \hat{b}, \hat{p})$, for $L - a.e.h$; *iii)* $\int_{[0,1)} \hat{A}(h, \omega, t)dh = y(t)$, $L \times L - a.e.(\omega, t)$. *iv)* $F(\hat{x}(t), \hat{y}(t)) \geq \hat{a}(t)$, $L - a.e.t.$ We exploit propositions 18 and 19. Hence, for given initial condition, we take a competitive equilibrium allocation of the concavified economy and we show that the same prices support a $CPSE$ with equivalent allocation. This result implies, by propositions 18 and 19, that any efficient (lottery) allocation of the large economy can be equivalently supported by a $CPSE$.

The following observation, although not necessary (see for instance [5]), greatly simplifies the argument. Consider the set $\Xi = \{(y,u) : y \in Y$ and $u = U(y)\} \subset R^{C+1}\}$. Let $CH(\Xi) \subset R^{C+1}$ be the convex hull of Ξ. Consider an equilibrium allocation of the "concavified" economy. For each t, by the definition of least concavity, $(\hat{y}(t), V(\hat{y}(t))) \in CH(\Xi)$. Then, Caratheodory's theorem implies that there exists at most $C+2$ consumption bundles $(\hat{y}_1(t), ..., \hat{y}_{C+2}(t))$ and $\hat{\beta}(t) \in \Delta(C+2)$ such that $\Sigma_\kappa \hat{\beta}_\kappa(t)\hat{y}_\kappa(t) = \hat{y}(t)$ and $\Sigma_\kappa \hat{\beta}_\kappa(t) U(\hat{y}_\kappa(t)) = V(\hat{y}(t))$. Hence, when we analyze efficient (lottery) allocations of the large economy there is no loss of generality in restricting attention to allocations $\hat{\beta}(t)$ with support consisting of at most $C+2$ points.

Proposition 20 *Let $(\hat{q}, \hat{b}, \hat{p}; \hat{x}, \hat{a}, \hat{y})$ be an intertemporal equilibrium of the concavified economy (V, F, x_0). Then, there is a sunspot allocation $\hat{A}$, with*

$$\int \hat{A}(h,\omega,t)d\omega = \hat{y}(t),$$

and $\int_{[0,1)} U(\hat{A}(h,\omega,t))d\omega = V(\hat{y}(t))$, for $L \times L - a.e.(h,t)$, such that the vector $(\hat{q}, \hat{b}, \hat{p}; \hat{x}, \hat{a}, \hat{y}, \hat{A})$ is an intertemporal CPSE of the large economy.

Proof By proposition 10 and corollary 11, at $(\hat{q}, \hat{b}, \hat{p})$ households maximize utility by selecting the capital and investment trajectories $(\hat{x}, \hat{a})$ and an individual sunspot allocation $\hat{y}^*$ such that

$$\int_{[0,1)} U(\hat{y}^*(t,\omega))d\omega = V(\hat{y}(t))$$

and $\int_{[0,1)} \hat{y}^*(t,\omega)d\omega = \hat{y}(t)$. Hence, we just have to show, that there exists a function $\hat{A}(.)$ such that $\hat{A}(h) = \hat{y}^*$, for all h, and $\int_{[0,1)} A(h,\omega,t)dh = \hat{y}(t)$, $L \times L - a.e.(t,\omega)$. By Caratheodory's theorem, for each $\hat{y}(t)$ there exist $\hat{\beta}(t) \in \Delta(C+2)$ such that $U(y(\hat{\beta}(t))) = V(\hat{y}(t))$ and $y(\hat{\beta}(t)) = \hat{y}(t)$. Denote by $\hat{y}_\kappa(t)$, $\kappa = 1, ..., C+2$, the points in the support of $\hat{\beta}(t)$. Divide, for each t, the set of sunspots $[0,1)$ in the $C+2$ disjoint intervals, $I_\kappa(t) = [\Sigma_{j=0}^{\kappa-1}\hat{\beta}_j(t), \Sigma_{j=0}^{\kappa}\hat{\beta}_j(t))$, with the convention that $\hat{\beta}_0(t) = 0$. Let $\hat{+}$ denote addition modulo 1. Then $\hat{A}$ is defined as:

$$\hat{A}(h,\omega,t) = \hat{y}_\kappa(t),\ if\ \omega\hat{+}h \in I_\kappa(t).$$

Evidently, $\hat{A}(h) = \hat{y}^*$, for all h, and $\int_{[0,1)} \hat{A}(h,\omega,t)dh = \hat{y}(t)$, since $L\{h : A(h,\omega,t) = y_\kappa(t)\} = L\{h : h\hat{+}\omega \in I_\kappa(t)\} = \hat{\beta}_\kappa(t)$, for each pair (κ, t). ∎

8 Indeterminacy of static sunspot equilibria.

We construct an example of a large static economy with a countable infinity of sunspot equilibrium allocations, which are Pareto ranked. Once again, inefficient sunspot equilibria of the large static economy can never be stationary sunspot equilibria of the large economy. However, there is a significant difference between the example **??** and the present one, which justifies its construction. The static economy with a representative household bears no similarity with any finite dimensional Arrow-Debreu economy. This is particularly evident when equilibrium calls for random consumption allocations. The latter maps into randomness of the investment allocations, $a = F(x, y)$. Feasibility requires that investment be equal to zero in average. It is this requirement that makes the representative household, static economy very different than anything else.

The situation is quite different for a large economy. Randomness in individual consumption allocations does not translate (at efficient allocations) into randomness of investment allocations. When pro capita output is deterministic, feasibility in the static large economy is identical to feasibility in any finite dimensional Arrow-Debreu economy with a large number of identical households. In particular, it is immediate to show that for specific choices of the map F, the equilibrium of the large static economy is an equilibrium of a large pure exchange economy (with identical individuals) with the same preferences and endowments equal to the efficient pro capita aggregate consumption of the static economy. We make sure that this feature is present in our example. Hence, the Pareto ranked multiple equilibria of the large static economy are equilibria of the "isomorphic" pure exchange economy.

8.1 The economy

There is a unique consumption good and a unique capital good. The technology is $F(x, y) = 1/2(f(x) - y^2)$. f is chosen so that for $x^* = \arg\max(f(x)/2) - rx$, $f(x^*) = 1$. We identify the space of sunspots with a set of J points having identical probability $1/J$ for some integer $J > 1$. The prices of the consumption good are restricted to be sunspot invariant and normalized to one. The form of the technology F and the restrictions on sunspots and prices immediately imply that $q = 1$ and the output of the firm is equal to 1. Hence, the economy is isomorphic to a standard pure exchange economy populated by identical households with one unit of endowment of the only existing good. There is a unique consumption good and a continuum of identical households, with generic index h in the space $\{[0, 1), B, L\}$. The utility function U is piecewise linear and non decreasing. [3] More precisely the function U is defined by:

$$U = \begin{pmatrix} 0, \text{ for } y \in [0, y_*], \\ \alpha(y - y_*), \text{ for } y \in [y_*, y_b], \\ \alpha(y_b - y_*) + \gamma(y - y_b), \text{ for } y > y_b. \end{pmatrix}$$

[3] It should be evident from the analysis that, at the cost of heavier computations, all the conclusions can be generalized to strictly increasing and non concave functions.

We assume:

$H1)\ y_* < 1 < y_b,\ H2)\ 1 > \delta = \alpha(y_b - y_*)/y_b > \gamma$ and $H\prime 2)\ \alpha(1 - y_*) - \gamma > 0.$

$H1$ allows for the existence of sunspot equilibria. By $H2$ there is a least concave function V associated to U. $H2$ and $H'2$ are not essential, but they simplify the computations. The least concave function associated with U is:

$$V = \begin{pmatrix} \delta y, \text{ for } y \in [0, y_b], \\ \alpha(y_b - y_*) + \gamma(y - y_b), \text{ for } y > y_b. \end{pmatrix}$$

There is a complete set of contingent commodity markets, where individuals buy contingent commodity bundle $y^J = (y_1, ..., y_J)$ at prices $p_j = 1$, for all j. In the next two sections we drop the subscript h. Since states are equiprobable, the programming problem of the households is:

$$(H^J) \qquad \max_{y^J} \Sigma_j U(y^j), \text{ subject to } \sum_j (y_j - 1) = 0.$$

Let $y_h(J)$ be the set of optimal solutions to (H^J). An allocation A^J is a measurable map from $[0,1)$ to R^J_+. $A^J(h)$, $h \in [0,1)$, is the vector of contingent commodities of individual h prescribed by the allocation map A^J.

An allocation A^J is feasible if

$$\int_{[0,1]} A^J(h,j)L(dh) = 1, \text{ for all } j = 1, ..., J.$$

Given J, a sunspot equilibrium is a feasible allocation such that $A(h) \in y(J)$, for $L - a.e.h.$

8.2 Individual Optimization Problem

Facing several equiprobable states and sunspot invariant prices, households optimize by selecting few consumption bundles and making endogenous their probabilities by associating an endogenous number of states to each distinct consumption bundle. This idea is formalized by the next lemma the proof of which is in the appendix. Let $S(y^J) = \{y \in R_+ : y_j = y, \text{ for some } j\}$, $y^J \in y(J)$.

Lemma 21 *$\#S(y^J) \leq 2$, for all J and some $y^J \in y(J)$.*

Because of Lemma 21, in order to get insights into the optimal solution to (H^J), we can study, first, a fictitious and simple programming problem with two states of uncertainty and with $\sigma \in (0,1)$ denoting the probability of the first state. The two prices are sunspot invariant and equal to1. The household solves:

$$(H^\sigma)\ \max \sigma U(y_1) + (1-\sigma)U(y_2), \ \textit{subject to } \sigma y_1 + (1-\sigma) y_2 \leq 1$$

The set of optimal solution to (H^σ) is denoted by $\hat{y}(\sigma) = (y_1, y_2)(\sigma)$. Evidently, $(y_1, y_2) \in \hat{y}(\sigma)$ if and only if $(y_1, y_2) \in \hat{y}(1-\sigma)$. Hence, we just analyze the problem for $\sigma \in [1/2, 1)$.

Let σ^* be such that

$$(1-\sigma^*)\hat{y}_b = 1$$

By $H1$, $\sigma^* \in (0,1)$ and since $(1-\sigma^*)U(y_b) = V((1-\sigma^*)y_b)$, $\hat{y}(\sigma^*) = (0, y_b)$. Given the symmetry of the problem, we assume, without loss of generality, that

$$\sigma^* > 1/2.$$

Let

$$\hat{y}_H(\sigma) = (0, \frac{1}{1-\sigma})1,\ U_H(\sigma) = (1-\sigma)U(\frac{1}{1-\sigma}),\ \text{and } \bar{\sigma} = \frac{(\alpha-\gamma)(y_b-1)}{y_b(\alpha-\gamma)--\alpha y_*} \in (0,1).$$

By $H2$ and $H2'$, $\bar{\sigma} \in (\sigma^*, 1)$. The optimal solution and the value, $W(\sigma)$, of the programming problem, (H^σ), are characterized by the next lemma whose proof is postponed to the appendix.

Lemma 22 *The optimal solution to (H^σ) is equal to*

$$\hat{y}(\sigma) = \begin{pmatrix} \hat{y}_H(\sigma),\ \textit{for } \sigma \in [1/2, \bar{\sigma}), \\ \hat{y}(\bar{\sigma}) = \hat{y}_H(\bar{\sigma}) \cup \bar{1},\ \textit{for } \sigma = \bar{\sigma}, \\ \hat{y}(\sigma) = \bar{1},\ \textit{for } \sigma \in (\bar{\sigma}, 1]. \end{pmatrix}$$

The value to H^σ, $W(\sigma)$, is equal to $U_H(\sigma)$ and strictly increasing in $[1/2, \sigma^*]$, equal to $U_H(\sigma)$ and strictly decreasing in $[\sigma^*, \bar{\sigma})$, and constant and equal to $U(1)$, for $\sigma \in [\bar{\sigma}, 1]$. Furthermore, $W(\sigma^*) > W(\sigma) > U(1)$, for $\sigma \in [1/2, \sigma^*)$.

Lemma 22 implicitly characterizes the optimal solution to the programming problem H^J. Given an arbitrary J, let $n(J) \in \arg\max_{\frac{J}{2} \le n \le J} W(\frac{n}{J})$. (Remember that, by the symmetry in σ of the programming problem H^σ, $W(\frac{n}{J}) = W(\frac{J-n}{J})$. Hence, the constraint $\frac{J}{2} \le n \le J$ is without loss of generality.) By Lemma 22, $\frac{n(J)}{J} \in [\bar{\sigma}, 1]$ only if $\frac{n}{J} \notin [1/2, \bar{\sigma})$, for all $\frac{J}{2} \le n \le J$. Evidently, if J is even or sufficiently high, $\frac{n(J)}{J} \in [1/2, \bar{\sigma})$. Let J^* be the smallest odd integer such that $\frac{n}{J} \in (1/2, \bar{\sigma})$ for some $n \le J$. Then:

Lemma 23 *For $J \ge J^*$, $y(J) = \hat{y}_H(\frac{n(J)}{J})$.*

8.3 Sunspot Equilibria

We construct a family of sunspot equilibrium allocations indexed by J. The construction is basically identical to the one used in the proof of Proposition 20.

For $J > J^*$, partition the interval $[0,1]$ in J subintervals, $I_j = [j-1/J, j/J)$, $j = 1, ..., J$. For each j and h, let $h\hat{+}(\frac{j}{J})$ be an addition modulo 1. Then A^J is defined as:

$A^J(h,j) = y_H(\frac{n(J)}{J})$ if $h\hat{+}(\frac{j}{J}) > (\frac{n(J)}{J})$, while $A(h,j) = 0$ if $h\hat{+}(\frac{j}{J}) \leq (\frac{n(J)}{J})$.

The allocation map A^J is feasible and $A^J(h) \in y(J)$, for all h. Furthermore, $(\frac{1}{J})\Sigma_j U^J(A(h,j)) = W(\frac{n(J)}{J})$.

Proposition 24 *There exists a countable infinity of Pareto ranked sunspot equilibria.*

Proof: It suffices to show that there exists a sequence of integers J_k such that $W(\frac{n(J_k)}{J_k})$ is monotonically increasing and $W(\frac{n_{J_k}}{J_k}) \to W(\sigma^*)$. For $J > J^*$, let $n^-(J) \in \arg\max_{\frac{1}{2} \leq \frac{n}{J} \leq \sigma^*}(\frac{n}{J} - \sigma^*)$ and $n^+(J) = \arg\max_{\sigma^* \leq \frac{n}{J} < \bar{\sigma}}(\frac{n}{J} - \sigma^*)$. Evidently, $n^-(J) = n^+(J) - 1$ and $\frac{n(J)}{J} \in \arg\max_{n \in \{n^-(J), n^+(J)\}} W(\frac{n}{J})$.

If σ^* is an irrational number, set $J_k = k$. If σ^* is a rational number, let (n^*, N^*), $n^* \leq N^*$, be the pair of natural numbers satisfying a) $\sigma^* = \frac{n^*}{N^*}$, and b) $N + n > N^* + n^*$, for each pair of natural numbers $(n, N) \neq (n^*, N^*)$ such that $\sigma^* = \frac{n}{N}$. If N^* is even, consider a sequence $J_k = 2k + 1$, and if it is odd $J_k = (2)^k$. In both cases, $\sigma^* - \frac{n(J_k)}{J} \neq 0$, for all k, and $\lim_{k \to \infty}(\sigma^* - \frac{n(J_k)}{J}) = 0$. By using if necessary subsequences, we get the claim. ■

References

[1] Cass, D. (1989), "Sunspots and incomplete financial markets: The leading example," in G. Feiwell, ed., *"The economics of imperfect competition and employment, Joan Robinson and beyond,"* (Macmillan, London).

[2] Cass, D. and Polemarchakis, H, (1990), "Convexity and Sunspots: A Remark," *Journal of Economic Theory, 52, 433-439.*

[3] Cass, D. and Shell, K., (1983), "Do sunspot matter?", *Journal of Political Economy,* 91 (1983), 193-227.

[4] Garrat, R., (1995), "Decentralizing lottery allocations in markets with indivisible commodities," *Economic Theory, 5, 295-313.*

[5] Garrat, R., Keister, D., Qi, C. and Shell, K., (2003), "Equilibrium Prices When the Sunspot Variable is Continuous", *Journal of Economic Theory,* 107, 11-38.

[6] Garratt, R. and Keister, T., (2002) "A characterization of robust sunspot equilibria," *Journal of Economic Theory, 107, 136-144..*

[7] Goenka, A. and K. Shell, K., (1997) "Robustness of sunspot equilibria," *Economic Theory, 10, 79-98.*

[8] Kehoe, T., Levine, D. and Prescott, E., (2001), "Lotteries, Sunspots and Incentives Constraints", *Journal of Economic Theory, 107, 39-69.*

[9] Lancaster, K. L., (1966), “A New Approach to Consumer Theory”, *Journal of Political Economy*, 74, 2, 132-157.

[10] *Journal of Economic Theory,* (November 2002), Volume 107, Issue 1, Pages 1-157.

[11] Prescott, E. and Townsend, (1984), ”Pareto Optima and Competitive Equilibria in an Economy with Adverse Selection and Moral Hazard,” *Econometrica*, 52(1), 21-45

[12] Rustichini, A. and Siconolfi, P., (2003), " Non-convexities in preferences over commodities and the Lancaster’s model of characteristics," mimeo.

[13] Rustichini, A. and Siconolfi, P., (2002), "Growth with non-convex preferences," mimeo.

[14] A. Rustichini, A. and Siconolfi, P.(2002), "Large Economies with non-convex preferences," mimeo.

[15] Shell, K., (1987), "Sunspot Equilibrium," in *The New Palgrave: General Equilibrium,* Eatwell, J., Milgate, M., and Newman, P., eds. W.W. Norton.

[16] Shell, K., and Wright, R., (1993), "Indivisibilities, Lotteries and Sunspot Equilibria," *Economic Theory, 3, 1-17.*

[17] Valadier, M., (1993), "A course on Young Measures," mimeo.

9 Appendix.

Proof of Lemma 7 The argument is based on some properties of Young measures [17]. A Young measure on $R_+ \times Y$ is a positive measure τ such that for any Borel set $A \subset R_+$, $\tau(A \times Y) = L(A)$, where L is the Lebesgue measure. The narrow topology on the set of Young measures is defined by the duality of these measures with the Caratheodory’s integrands; equivalently a sequence of Young measures converges narrowly if and only if the inner product with any Caratheodory’s integrand converges.

Let $\hat{y} : R_+ \to Y$ be a measurable map. The unique Young measure associated to $\hat{y}$, ν, is such that for any function $\hat{U} : R_+ \times Y \to \bar{R}$ measurable and ≥ 0, $\int_{R_+ \times Y} \hat{U} d\nu = \int_{R_+} \hat{U}(t, \hat{y}(t))dt$. Since in our case, Y is compact, $\hat{U} = e^{-rt}U$ and U is continuous, the requirement $\hat{U} \geq 0$ is without loss of generality. A Young measure ν can be represented by its disintegration, which is a family $(\beta_t)_{t \in R_+}$ of probabilities over Y such that for any function $\hat{U} : R_+ \times Y \to \bar{R}$ measurable and non negative, $\int_{R_+ \times Y} \hat{U} d\nu_\omega = \int_{R_+} [\int_Y \hat{U}(t, y)\beta_t(dy)]dt$. If ν is associated to $\hat{y}$, $\beta_t = \delta_{\hat{y}(t)}$, for $\delta_{\hat{y}(t)}$ denoting the Dirac mass at $\hat{y}(t)$. The set of Young measures associated to functions is dense (in the narrow topology) in the set of Young measures. Thus, if (U) has a solution $\hat{y}^* = (\hat{y}^*(\omega, .))_{\omega \in [0,1)}$, the family of Young measures $(\nu_\omega)_{\omega\in[0,1)}$ associated to $\hat{y}$ and, equivalently, its disintegration $\hat{\beta}^*$, $\hat{\beta}^*(\omega,t) = \delta_{\hat{y}^*(\omega,t)}$ is an optimal solution to:

$$(U') \qquad \max_{\hat{\beta}(t)\in M_{+,\cdot}([0,1),\hat{B})} \int_{\Re_+} e^{-rt}(\int_{[0,1)} (U(\hat{\beta}(\omega,t))d\omega)dt$$

subject to

$$\int_{\Re_+} (\int_{[0,1)} \hat{p}(\omega,t)y(\hat{\beta}(\omega,t))d\omega)dt.$$

where $M_{+,\cdot}([0,1),\hat{B}) = \{\hat{\beta}(t) : \Omega \to M_{+,\cdot}(Y) : \beta(t) \text{ is } \{[0,1),\hat{B}\}-measurable\}$.

The argument is concluded by the next claim that creates an obvious parallelism between the programming problems $(U')$and (V).

An optimal solution to (U'), $\hat{\beta}^*$ and to (V), $\hat{y}^*$, are equivalent if $\int_\Omega V(\hat{y}^*(\omega,t))d\omega = \int_\Omega U(\hat{\beta}^*(\omega,t))d\omega$ and $\hat{y}^*(\omega,t) = y(\hat{\beta}(\omega,t))$, for $L\times L-a.e.(\omega,t)$.

Claim 25 *Optimal solution to (U') and to (V) are equivalent.*

Proof This is a trivial consequence of the linearity (in contingent commodities) of $\hat{p}$ and the definition of V. ■

Proof of Lemma 21 This is consequence of Jensen's inequality. More precisely, the utility function U is concave in $[0,y_*]$ and in $[y_*,\infty)$. For each $y^J \in y^J(J)$, let $y^{J_1} = (y_1,...,y_{J_1})$ be the vector of components of y^J contained in the interval $[0,y_*]$, i.e., $y^{J_1} \in [0,y_*]^{J_1}$. There is no loss of generality in assuming that the entries of y^{J_1} are the first J_1 entries of y^J. Let $J_2 = J - J_1$. Evidently, by construction, $y^{J_2} = (y_{J_1+1},...,y_J) \in (y_*,\infty)^{J_2}$. Then:

$$\Sigma_j U(y_j) = J_1\Sigma_{j=1}^{J_1}\frac{U(y_j)}{J_1} + J_2\Sigma_{j=J_1+1}^{J}\frac{U(y_j)}{J_2} \leq$$

$$[(J_1U(\Sigma_{j=1}^{J_1}(y_j/J_1)) + J_2U(\Sigma_{j=J_1+1}^{J}(y_j/J_2)].$$

■

Proof of Lemma 22 Given the shape of U, for $\sigma \in [1/2,1)$, the optimal solutions to (U^σ) is either i) the constant bundle $\bar{1} = (1,1)$ and/or ii) if $\hat{y} \in \hat{y}(\sigma)$ and $\hat{y} \neq \bar{1}$, then $\min\{y_1,y_2\} = 0$ and $\sigma\hat{y}_1 + (1-\sigma)\hat{y}_2 = 1$.

There are just two candidates that satisfy *ii*). They are obtained by setting equal to 0 either the first or the second entry of the consumption bundle. The first candidate $\hat{y}_H(\sigma)$ has already been defined, the second, $\hat{y}_L(\sigma)$, is defined as follows:

$$\hat{y}_L(\sigma) = (\frac{1}{\sigma},0) \text{ and } U_L(\sigma) = \sigma U(\frac{1}{\sigma})$$

Let $\Phi(\sigma) = U_H(\sigma) - U_L(\sigma)$. For $\sigma \in [\sigma^*,1)$, $\frac{1}{1-\sigma} \geq y_b > \frac{1}{\sigma}$, and hence:

$$\Phi(\sigma) = (1-\sigma)[(\alpha-\gamma)y_b - \gamma y_*] + \gamma - \alpha(1-\sigma y_*).$$

Then:

$$\Phi(\sigma^*) = (\alpha\sigma^* - (1-\sigma^*)\gamma)y_*$$
$$\Phi(1) = \gamma - \alpha(1-y_*).$$

Since $\sigma^* > 1/2$ and $\alpha > \gamma$, $\Phi(\sigma^*) > 0$. Furthermore, by $U3$, $\Phi(1) > 0$, Hence:

$$U_H(\sigma) > U_L(\sigma),\ for\ \sigma \in [\sigma^*, 1),$$

For $\sigma \in (1/2, \sigma^*]$, $\frac{1}{1-\sigma} > \frac{1}{\sigma} > y_*$ and, by direct computation:

$$\Phi(\sigma) = \alpha(1 + (2\sigma - 1)y_*) > 0$$

Hence, $U_H(\sigma) > U_L(\sigma)$, for all $\sigma \in [1/2, 1)$. Furthermore:

$$U_H(\sigma) - U(1) = \left\{ \begin{array}{c} \alpha\sigma y_*, \text{ for } \sigma \in [1/2, \sigma^*) \\ (\alpha - \gamma)((1-\sigma)y_b - 1) + \alpha\sigma y_*, \text{ for } \sigma \in [\sigma^*, 1) \end{array} \right\}.$$

Therefore, by trivial computations, $U_H(\bar{\sigma}) = U(1)$, $W(\sigma) = U_H(\sigma)$, for $\sigma \in [1/2, \bar{\sigma})$, while $W(\sigma) = U(1)$, $\sigma \in [\bar{\sigma}, 1)$. Finally, by taking into account assumption $H2$),

$$dW(\sigma)/d\sigma = \left\{ \begin{array}{c} \alpha y_* > 0, \text{ for } \sigma \in [1/2, \sigma^*) \\ -(\alpha - \gamma)y_b + \alpha y_* < 0, \text{ for } \sigma \in (\sigma^*, \bar{\sigma}) \\ 0, \text{ for } \sigma \in [\bar{\sigma}, 1) \end{array} \right\}. \qquad \blacksquare$$